European Society of Computational Methods in Sciences, Engineering and Technology (ESCMSET)

Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM)
vol. 7, no. 3-4, 2012, pp. 59-71
ISSN 1790–8140

Piecewise Linear Finite Element Approximation of a Generalized Stokes System Related to Viscolelastic Flow[1]

V. Ruas [2], A.P. Brasil [3], J.H. Carneiro de Araujo [4]

Received 14 December, 2010; accepted in revised form 25 April, 2012

Abstract: A three-field finite element scheme designed for solving systems of partial differential equations governing stationary viscoelastic flows is studied. It is based on the simulation of a time-dependent behavior. Once a classical time-discretization is performed, the resulting three-field system of equations allows for a stable approximation of velocity, pressure and extra stress tensor, by means of continuous piecewise linear finite elements, in both two and three dimension space. This is proved to hold for the linearized form of the system. An advantage of the new formulation is the fact that it implicitly provides an algorithm for the iterative resolution of system non-linearities. Convergence in an appropriate sense applying to these three flow fields is demonstrated.

Keywords: Finite element, Multi-field methods, Stokes system, Viscoelastic flows

Mathematics Subject Classification: 65N30, 76M10, 76D05

1 Introduction

The numerical simulation of the flow of viscoelastic liquids is known to be a delicate problem in many respects. First of all, the models most frequently in use, involve three strongly coupled fields, namely, the velocity $\mathbf{u}$, the pressure p and the extra stress tensor σ. Furthermore the highly non linear system of partial differential equations that govern this kind of flows may change of type, according to different parameters or flow conditions. Nevertheless in the past two decades a lot of progress has been accomplished in deriving numerical methodology, in order to overcome such difficulties.

As far as multi-field finite element methods suitable for treating this class of problems in a reliable way are concerned, the work of FORTIN and collaborators (see e.g. [11]) incorporating a fourth field, namely, the strain rate tensor, lies among the most outstanding contributions in this direction. In particular it significantly propelled the numerical simulation of viscolelastic

[1]Published electronically December 15, 2012

[2]Corresponding author. UPMC-Université Paris 6, UMR 7190, Institut Jean le Rond d'Alembert, France & Visiting Professor at Department of Mechanical Engineering, PUC-Rio (Pontifícia Universidade Católica do Rio de Janeiro), Brazil. E-mail: vitoriano.ruas@upmc.fr

[3]Department of Mechanical Engineering, Faculty of Technology, Universidade de Brasília, Brazil. E-mail: brasiljr@unb.br

[4]Department and Graduate School of Computer Science, Universidade Federal Fluminense, Niterói, Brazil. E-mail: jhca@ic.uff.br

fluid flow in three-dimension space, which became more widespread in the 2000's (cf. [2]). As for the numerical analysis of finite element methods for the complete set of equations governing viscoelastic flows the contribution of BARANGER and SANDRI [3] is a main reference. Confining their analysis to the linearized case, the first and third authors themselves attempted to bring about valid alternatives to study this class of problems mostly in the two-dimensional case, through drastic reductions of the number of degrees of freedom necessary to obtain reliable approximations (cf. [21]), as compared to other methods in use of the same order (cf. [17]). However in the framework of three-dimensional flows, such approach is not satisfactory, since the final number of degrees of freedom remains excessively high anyway. On the other hand, handling system non linearities is another challenge that any numerical methodology must be able to take up efficiently, especially in the context of the problem under consideration.

In this work the authors present a new three-field formulation of the system of equations that govern stationary viscoelastic flow, aimed at overcoming simultaneously both difficulties above, either in two- or in three-dimension space. More specifically it is shown that the use of classical continuous piecewise linear interpolations of $\mathbf{u}$, p and σ, combined with a suitable variational formulation, leads to stable and accurate discrete counterparts of an algorithm for the iterative treatment of a generalized Stokes system, obtained by linearizing viscolelastic flow equations. Those ingredients are based on the time integration of the underlying time-dependent system, by means of a splitting algorithm inspired by the ideas already exploited by Goldberg & Ruas in [15] for the time integration of the incompressible Navier-Stokes equations.

2 Maxwell Flow Equations

Although the technique to be developped hereafter extends in a straightforward manner to the case of a wide spectrum of viscolelastic constitutive laws, for the sake of simplicity, we consider as a model the case of Maxwell fluids.

Let then Ω be a bounded domain of R^N, $N = 2$ or 3, with boundary $\partial\Omega$. Under the action of volumetric forces $\mathbf{f}$, we consider the evolution in time t, of the flow in Ω of a viscoelastic liquid obeying a constitutive law of the differential type. Throughout this work we assume that the velocity of the liquid is prescribed on $\partial\Omega$, say $\mathbf{u} = \mathbf{g}$. Moreover without any loss of essential aspects, just to simplify the presentation, we consider a constitutive law of the upper convected type, which relates the extra stress tensor to the velocity in the following manner:

$$\sigma + \lambda \left[\frac{\partial \sigma}{\partial t} + (\mathbf{u} \cdot \nabla)\, \sigma - (\nabla \mathbf{u})\sigma - \sigma(\nabla \mathbf{u})^T \right] = 2\eta D(\mathbf{u}). \tag{1}$$

In (1) λ is the stress relaxation time of the liquid and η is its reference viscosity, both assumed to be constant; ∇ represents the gradient of a scalar or a vector valued function and $D(\mathbf{u})$ denotes the strain rate tensor, i.e., $D(\mathbf{u}) := \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right]$.

Then from a given state at time $t = 0$, that is, given a solenoidal velocity $\mathbf{u}^0$ and an extra stress σ^0, for $t > 0$, in addition to the law (1), the flow is governed by the following system:

$$\left.\begin{array}{rcl} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} - \nabla \cdot \sigma + \nabla p & = & \mathbf{f} \\ \nabla \cdot \mathbf{u} & = & 0 \end{array}\right\} \text{in } \Omega \times (0, \infty) \tag{2}$$

where the density of the liquid is assumed to be equal to one.

In this work we will be concerned about the search of steady state solutions. Therefore we shall further assume in all the sequel, that both $\mathbf{f}$ and $\mathbf{g}$ are independent of t.

Now we consider the following semi-implicit discretization in time of system (1)-(2). Let $\Delta t > 0$ be a given time step, and $\mathbf{u}^n$, p^n and σ^n denote approximations of $\mathbf{u}(n\Delta t)$, $p(n\Delta t)$ and $\sigma(n\Delta t)$, respectively, for a strictly positive integer n. Starting from $\mathbf{u}^0$ and σ^0, and prescribing $\mathbf{u}^n = \mathbf{g}$ on $\partial\Omega$ for every n, $\mathbf{u}^n$, p^n and σ^n, for $n = 1, 2, \ldots$, are determined as the solution of the following system in Ω:

$$\left\{ \begin{array}{rcl} \dfrac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} + (\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{n-1} - \nabla \cdot \sigma^n + \nabla p^n & = & \mathbf{f} \\ \nabla \cdot \mathbf{u}^n & = & 0 \\ \sigma^n + \lambda \left[\dfrac{\sigma^n - \sigma^{n-1}}{\Delta t} + (\mathbf{u}^{n-1} \cdot \nabla)\sigma^{n-1} - (\nabla \mathbf{u}^{n-1})\sigma^{n-1} - \sigma^{n-1}(\nabla \mathbf{u}^{n-1})^T \right] & = & 2\eta D(\mathbf{u}^n) \end{array} \right. \tag{3}$$

As one can readily infer, (3) is a linear problem for every n. Actually assuming moderate velocities and velocity gradients, the non linear terms may be neglected. In this case we can legitimately linearize (1)-(2) into the system governing the very slow flow of a viscoelastic fluid of the Maxwell type. Actually for the sake of conciseness we introduce our methodology in the context of the following generalized Stokes system, derived from the linearization of the equations that govern the flow of a Maxwell viscolelastic liquid (cf. [17]), namely:

From a given state at time $t = 0$ defined by a given solenoidal velocity $\mathbf{u}^0$ and an extra stress tensor σ^0, for $t > 0$ find $p, \mathbf{u}, \sigma$ that solve the following system, with $\mathbf{u} = \mathbf{g}$ on $\partial\Omega \times (0, \infty)$:

$$\left. \begin{array}{rcl} \dfrac{\partial \mathbf{u}}{\partial t} - \nabla \cdot \sigma + \nabla p & = & \mathbf{f} \\ \nabla \cdot \mathbf{u} & = & 0 \\ \sigma + \lambda \dfrac{\partial \sigma}{\partial t} & = & 2\eta D(\mathbf{u}) \end{array} \right\} \text{in } \Omega \times (0, \infty). \tag{4}$$

3 Time discretization and splitting algorithm

Splitting algorithms based on projections onto spaces of solenoidal fields were first proposed by CHORIN [8] and TEMAM [24]. They have since proved to be an efficient tool to solve the incompressible Navier-Stokes equations. One of its main features is handling the two primitive variables, that is velocity and pressure, in an uncoupled manner. Another important advantage of this kind of approach is that, at least in some versions, it allows for the use of the simplest possible space discretizations for both variables without affecting numerical stability. This property was exploited by many authors (cf. CODINA & ZIENKIEWICZ [10] for example). However the main drawback of projection algorithms as reported by different authors (see e.g. RANNACHER [19]) remained a persistent numerical inconsistency in their versions most widely in use. This is especially true of those employing a pressure Poisson equation with unphysical Neumann boundary conditions. In GOLDBERG & RUAS [15] an alternative to this pressure solver aimed at overcoming this difficulty was first proposed. The basic idea was the computation of a post-processed pressure at each time step from the available velocity, by a least-square approach, using the momentum equation. The numerical results certified a considerable improvement of the thus corrected pressure, as compared to the one obtained in a classical way, at least for Reynolds numbers not very low. Indeed, the fact that the viscous term was systematically purged from the true boundary conditions for the corrected pressure equation, caused a more significant loss of accuracy the lower the Reynolds number was (cf. [15]). This is because second order derivatives are not computable with classical lagrangean finite elements. Although remedies for this problem were proposed and tested in works performed under the first author's guidance (see e.g. [18]), a persistent lack of accuracy in pressure computations was systematically reported. Notice that in [16] a modification of the above mentionned pressure correction technique was proposed, in order to circumvent such inconsistency

of projection algorithms. However the authors do not show that their approach allows for the use of finite element approximations violating the classical inf-sup condition (see e.g. [6]), such as continuous piecewise linear for both variables.
In this section we propose a new algorithm for solving both newtonian and non newtonian flow equations, in the $\mathbf{u}, p, \sigma$ formulation. Although this technique is described here only in the context of problem (4), its adaption to more general cases is straightforward, including for instance the Navier-Stokes equations, or yet turbulent flow with turbulent stress models. Indeed in the latter cases it suffices to take $\lambda = 0$, before incorporating non linear expressions or terms. It seems however that in the context of viscolastic flow the new approach appears to be the most promising, since in this case the use of a three-field formulation is mandatory.

We have mainly dealt with an explicit splitting algorithm for the time integration or the iterative solution of system (4). However before presenting it we consider the underlying implicit discretization in time of (4).

Let $\Delta t > 0$ be a given time step. Then starting from $\mathbf{u}^0$ and σ^0, for $n = 1, 2, \ldots$, and prescribing $\mathbf{u}^n = \mathbf{g}$ on $\partial\Omega$ for every n, we determine approximations of $p(n\Delta t)$, $\mathbf{u}(n\Delta t)$ and $\sigma(n\Delta t)$, denoted by p^n, $\mathbf{u}^n$ and σ^n respectively, as the solution of the following problem:

$$\left.\begin{array}{rcl} \dfrac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} - \nabla \cdot \sigma^n + \nabla p^n & = & \mathbf{f} \\ \nabla \cdot \mathbf{u}^n & = & 0 \\ \sigma^n + \lambda \left(\dfrac{\sigma^n - \sigma^{n-1}}{\Delta t} \right) & = & 2\eta D(\mathbf{u}^n) \end{array}\right\} \text{ in } \Omega. \quad (5)$$

For the sake of simplicity we assume that Ω has suitable non restrictive regularity properties. Moreover we further assume that $\mathbf{f} \in [L^2(\Omega)]^N$, $\mathbf{g} \in [H^{3/2}(\partial\Omega)]^N$, $\mathbf{u}^0 \in [H^1(\Omega)]^N$ and $\sigma^0 \in [H^1(\Omega)]^{N \times N}$ (cf. [1]). Let also $< \cdot, \cdot >_{1/2, \partial\Omega}$ denote the duality product between $[H^{1/2}(\partial\Omega)]^N$ and $[H^{-1/2}(\partial\Omega)]^N$, $(\cdot, \cdot)$ and $\| \cdot \|$ denote the standard L^2-inner product and associated norm, respectively.

We will be mainly concerned about the convergence of sequences of approximations of p, $\mathbf{u}$ and σ in the sense of $L^2(\Omega)$. However referring to [14] for the definition of some spaces and notations used below, the natural spaces (resp. manifold) in which these three unknowns fields are searched for in our formulation are respectively.

$Q := \{q | \; q \in H^1(\Omega) \cap L^2_0(\Omega)\}$;
$\mathbf{V}^{\mathbf{g}} := \{\mathbf{v} | \; \mathbf{v} \in \mathbf{V}, \; , \mathbf{v} = \mathbf{g} \text{ on } \partial\Omega\}$, with $\mathbf{V} := [H^1(\Omega)]^N$;
$\Sigma := \{\tau | \; \tau \in [\mathbf{H}(\text{div}, \Omega)]^N, \; \tau^T = \tau\}$.

To begin with the convergence study we recall a result proven in [4]:

Theorem 3.1 *For every Δt and for every n problem (5) has a unique solution. Moreover the solution of (5) converges in the norm of $L^2(\Omega) \times [L^2(\Omega)]^N \times [L^2(\Omega)]^{N \times N}$, to the solution $(\bar{p}, \bar{\mathbf{u}}, \bar{\sigma})$ of the stationary counterpart of (4), namely*

$$\left\{ \begin{array}{lcll} -\nabla \cdot \bar{\sigma} + \nabla \bar{p} & = & \mathbf{f} & \text{in } \Omega; \\ \nabla \cdot \bar{\mathbf{u}} & = & 0 & \text{in } \Omega; \\ \bar{\sigma} & = & 2\eta D(\bar{\mathbf{u}}) & \text{in } \Omega; \\ \bar{\mathbf{u}} & = & \mathbf{g} & \text{on } \partial\Omega. \end{array} \right. \quad (6)$$

■

Remark 3.1 *We refer to [7] for a splitting algorithm to solve system (5) explicitly at every iteration. While such an algorithm is unlikely to generate converging sequence of approximations* *of $(p^n, \mathbf{u}^n, \sigma^n)$ in the continuous case, when applied to the discrete counterpart of (5) defined by replacing Q, $\mathbf{V}^{\mathbf{g}}$ and Σ by finite dimensional spaces (resp. manifold) Q_h, $\mathbf{V}_h^{\mathbf{g}}$ and Σ_h specified in the following section, it does lead to convergence. This is due to the validity of the following classical inverse inequality:*

$$\exists C_i > 0 \text{ depending on the dimension of } \Sigma_h \text{ s.t. } \parallel \nabla \cdot \sigma \parallel \leq C_i \parallel \tau \parallel \quad \forall \tau \in \Sigma_h. \tag{7}$$

Actually it is possible to establish that such a convergence is very fast, provided the dimension of Σ_h is not so high. ■

4 Space Discretization

Now we consider the following discrete analogue of (5). Henceforth we assume that Ω, $\mathbf{f}$ and $\mathbf{g}$ have regularity properties compatible with the regularity of the unknown fields required in the theorems that follow.

Let then T_h be a partition of Ω into N-simplices with maximum edge length equal to h. We assume that T_h satisfies the usual compatibility conditions for finite element meshes, and that it belongs to a quasi-uniform family of partitions. For every $K \in T_h$ we further denote by $P_1(K)$ the space of polynomials of degree less than or equal to one defined in K. In so doing we introduce the following spaces or manifolds associated with T_h:

$$S_h := \left\{ v \mid \; v \in C^0(\bar{\Omega}) \text{ and } v|_K \in P_1(K), \; \forall K \in T_h \right\},$$

$$\mathbf{V}_h := \{ \mathbf{v} \mid \forall i \; v_i \in S_h \}, \quad \mathbf{V}_h^0 := \mathbf{V}_h \cap \mathbf{V}^0, \text{ with } \mathbf{V}^0 := [H_0^1(\Omega)]^N,$$

$$\mathbf{V}_h^{\mathbf{g}} := \{ \mathbf{v} \in \mathbf{V}_h \mid \mathbf{v}(\wp) = \mathbf{g}(\wp) \; \forall \text{ vertex } \wp \text{ of } T_h \text{ on } \partial\Omega \},$$

$$Q_h := S_h \cap L_0^2(\Omega),$$

$$\Sigma_h := \left\{ \tau \mid \tau \in [S_h]^{N \times N}, \quad \tau^T = \tau \right\}.$$

Then letting $\mathbf{u}_h^0$ be the field of $\mathbf{V}_h^{\mathbf{g}}$ satisfying $\mathbf{u}_h^0(\wp) = \mathbf{u}^0(\wp)$, and σ_h^0 be the tensor of Σ_h satisfying $\sigma_h^0(\wp) = \sigma^0(\wp)$, for every vertex $\wp$ of T_h, we set the following problem to approximate (5) for every n, $n = 0, 1, 2, \ldots$

$$\left\{ \begin{array}{l} \text{Find } p_h^n \in Q_h, \mathbf{u}_h^n \in \mathbf{V}_h^{\mathbf{g}}, \text{ and } \sigma_h^n \in \Sigma_h \text{ such that} \\ \qquad (\nabla p_h^n - \nabla \cdot \sigma_h^n, \nabla q) = (\mathbf{f}, \nabla q) + \dfrac{1}{\Delta t} \left(\mathbf{u}_h^{n-1}, \nabla q \right), \;\; \forall q \in Q_h, \\ \qquad (\mathbf{u}_h^n - \Delta t (\nabla \cdot \sigma_h^n - \nabla p_h^n), \mathbf{v}) = \left(\mathbf{u}_h^{n-1} + \Delta t \mathbf{f}, \mathbf{v} \right), \;\; \forall \mathbf{v} \in \mathbf{V}_h^0, \\ \dfrac{\Delta t + \lambda}{2\eta} (\sigma_h^n, \tau) + \Delta t^2 (\nabla \cdot \sigma_h^n - \nabla p_h^n, \nabla \cdot \tau) = \dfrac{\lambda}{2\eta} \left(\sigma_h^{n-1}, \tau \right) - \\ \qquad \Delta t^2 (\mathbf{f}, \nabla \cdot \tau) - \Delta t \left(\mathbf{u}_h^{n-1}, \nabla \cdot \tau \right) + \Delta t < \mathbf{g}, \tau \vec{\nu} >_{1/2, \partial\Omega} \quad \forall \tau \in \Sigma_h. \end{array} \right. \tag{8}$$

Note that the third equation of (8) incorporates a term that stems from the momentum equation tested with $\nabla \cdot \tau$. It adds positiveness and in this sense it plays a stabilizing role, similarly to previous works like [12] and [13].

For problem (8) the following result holds.

Proposition 4.1 *Problem (8) has a unique solution for every Δt and every n (cf. [4]).* ■

Let us now consider the following weak form for problem (5). First we set

$$a\left((p,\mathbf{u},\sigma),(q,\mathbf{v},\tau)\right) := \Delta t^2\left(\nabla p - \nabla\cdot\sigma,\, \nabla q\right) + \Delta t\left(\mathbf{u},\, \nabla\cdot\tau - \nabla q\right) + \left(\mathbf{u} - \Delta t\left(\nabla\cdot\sigma - \nabla p\right),\, \mathbf{v}\right) + \frac{\Delta t + \lambda}{2\eta}\left(\sigma,\, \tau\right) + \Delta t^2\left(\nabla\cdot\sigma - \nabla p,\, \nabla\cdot\tau\right) \quad (9)$$

together with

$$L\left((q,\mathbf{v},\tau)\right) = \Delta t^2\left(\mathbf{f},\, \nabla q - \nabla\cdot\tau\right) + \Delta t < \mathbf{g},\, (\tau - Iq)\vec{\nu} >_{1/2,\partial\Omega} + \Delta t\left(\mathbf{f},\, \mathbf{v}\right) + \left(\mathbf{u}^{n-1},\, \mathbf{v}\right) + \Delta t\left(\mathbf{u}^{n-1},\, \nabla q - \nabla\cdot\tau\right) + \frac{\lambda}{2\eta}\left(\sigma^{n-1},\, \tau\right).$$

and we write (5) in the following equivalent variational form

$$\begin{cases} \text{Find } (p^n,\mathbf{u}^n,\sigma^n) \in Q\times\mathbf{V}^{\mathbf{g}}\times\Sigma \text{ such that} \\ a\left((p^n,\mathbf{u}^n,\sigma^n),(q,\mathbf{v},\tau)\right) = L\left((q,\mathbf{v},\tau)\right) \quad \forall\,(q,\mathbf{v},\tau)\in Q\times\mathbf{V}^0\times\Sigma \end{cases} \quad (10)$$

where a is defined in (9). First we prove the following convergence result:

Theorem 4.1 *For every Δt the solution of (8) converges to the solution of (10) in the norm of $L^2(\Omega)\times[L^2(\Omega)]^N\times[L^2(\Omega)]^{N\times N}$ as h goes to 0, provided for every n the solution of (10) belongs to $H^2(\Omega)\times[H^1(\Omega)]^N\times[H^2(\Omega)]^{N\times N}$.*

Proof. First we note that, for all $(p,\mathbf{u},\sigma,)$ and $(q,\mathbf{v},\tau)$ in $Q\times\mathbf{V}\times\Sigma$,

$$a\left((p,\mathbf{u},\sigma,),(q,\mathbf{v},\tau)\right) \le 2\parallel (p,\mathbf{u},\sigma) \parallel_{\Im}\parallel (q,\mathbf{v},\tau) \parallel_{\Im}.$$

where the norm $\parallel\cdot\parallel_{\Im}$ of $Q\times\mathbf{V}\times\Sigma$ is defined by:

$$\parallel (q,\mathbf{v},\tau) \parallel_{\Im}^2 := \frac{\lambda+\Delta t}{2\eta}\parallel \tau \parallel^2 + \Delta t^2\parallel \nabla\cdot\tau - \nabla q \parallel^2 + \parallel \mathbf{v} \parallel^2.$$

Thus from classical error bounds (see e.g. [9] and [23]), there exists a constant C such that

$$\parallel (p^n - p_h^n, \mathbf{u}^n - \mathbf{u}_h^n, \sigma^n - \sigma_h^n) \parallel_{\Im} \le C\left[\inf_{(q,\mathbf{v},\tau)\,\in\, Q_h\times\mathbf{V}_h^{\mathbf{g}}\times\Sigma_h} \parallel (p^n - q, \mathbf{u}^n - \mathbf{v}, \sigma^n - \tau) \parallel_{\Im} + \sup_{\begin{cases}(q,\mathbf{v},\tau) \in Q_h\times\mathbf{V}_h^0\times\Sigma_h \\ (q,\mathbf{v},\tau) \neq (0,\mathbf{0},\mathcal{O})\end{cases}} \frac{|\left(L_h - L\right)\left((q,\mathbf{v},\tau)\right)|}{\parallel (q,\mathbf{v},\tau) \parallel_{\Im}} \right]$$

On the other hand we know from standard estimates that there exists another constant C, independent of h, such that (cf. [9])

$$\inf_{(q,\mathbf{v},\tau)\,\in\, Q_h\times\mathbf{V}_h^{\mathbf{g}}\times\Sigma_h} \parallel (p^n - q, \mathbf{u}^n - \mathbf{v}, \sigma^n - \tau) \parallel_{\Im} \le Ch\left(\parallel \nabla\mathbf{u}^n \parallel + \parallel \nabla(\nabla\sigma^n) \parallel + \parallel \nabla(\nabla p^n) \parallel\right).$$

Moreover

$$\left(L_h - L\right)\left((q,\mathbf{v},\tau)\right) = \left(\mathbf{u}_h^{n-1} - \mathbf{u}^{n-1},\, \mathbf{v} + \Delta t\left(\nabla q - \nabla\cdot\tau\right)\right) + \frac{\lambda}{2\eta}\left(\sigma_h^{n-1} - \sigma^{n-1},\, \tau\right).$$

Therefore,

$$\sup_{\left\{\begin{array}{lcl}(q,\mathbf{v},\tau) & \in & Q_h\times\mathbf{V}_h^0\times\Sigma_h\\ (q,\mathbf{v},\tau) & \neq & (0,\mathbf{0},\mathcal{O})\end{array}\right.} \frac{|\left(L_h-L\right)\left((q,\mathbf{v},\tau)\right)|}{\|\ (q,\mathbf{v},\tau)\ \|_{\Im}} \leq \|\ \mathbf{u}_h^{n-1}-\mathbf{u}^{n-1}\ \| + \sqrt{\frac{\lambda}{2\eta}}\ \|\ \sigma_h^{n-1}-\sigma^{n-1}\ \|.$$

Now we note that, according to our assumptions there exists a constant $C(0)$, independent of h, such that

$$\|\ \mathbf{u}^0-\mathbf{u}_h^0\ \| + \|\ \sigma^0-\sigma_h^0\ \| \leq\ C(0)h\left(\|\ \nabla\mathbf{u}^0\ \| + \|\ \nabla(\nabla\sigma^0)\ \|\right).$$

Thus we have for a suitable constant $C(1)$, independent of h:

$$\|\ \mathbf{u}^1-\mathbf{u}_h^1\ \| + \|\ \sigma^1-\sigma_h^1\ \| \leq C(1)h\max_{0\leq i\leq 1}\left\{\|\ \nabla\mathbf{u}^i\ \| + \|\ \nabla(\nabla\sigma^i)\ \| + \|\ \nabla(\nabla p^i)\ \|\right\}.$$

Thus we recursively establish that there exist constants $C(n)$, independent of h, such that, for every n we have

$$\|\ \mathbf{u}^n-\mathbf{u}_h^n\ \| + \|\ \sigma^n-\sigma_h^n\ \| \leq C(n)h\max_{0\leq i\leq n}\left\{\|\ \nabla\mathbf{u}^i\ \| + \|\ \nabla(\nabla\sigma^i)\ \| + \|\ \nabla(\nabla p^i)\ \|\right\}. \tag{11}$$

As for the convergence of p_h^n to p^n we use here again a duality argument like in Theorem 3.1 (cf. [4]), that is

$$\|\ p^n-p_h^n\ \| \leq C \sup_{\left\{\begin{array}{lcl}\mathbf{v} & \in & \mathbf{V}^0\\ \mathbf{v} & \neq & \mathbf{0}\end{array}\right.} \frac{(\nabla p^n-\nabla p_h^n,\mathbf{v})}{\|\ \nabla\mathbf{v}\ \|} \leq$$

$$C\left\{\sup_{\left\{\begin{array}{lcl}\mathbf{v} & \in & \mathbf{V}^0\\ \mathbf{v} & \neq & \mathbf{0}\end{array}\right.} \frac{\Delta t\left(\sigma_h^{n+1}-\sigma^n,\nabla\mathbf{v}\right) + \left(\mathbf{u}_h^n-\mathbf{u}^n,\mathbf{v}\right) + \left(\mathbf{u}^{n-1}-\mathbf{u}_h^{n-1},\mathbf{v}\right)}{\Delta t\ \|\ \nabla\mathbf{v}\ \|} + \right.$$

$$\left.\sup_{\left\{\begin{array}{lcl}\mathbf{v} & \in & \mathbf{V}^0\\ \mathbf{v} & \neq & \mathbf{0}\end{array}\right.} \left[\frac{\left(\nabla\left(p^n-p_h^n\right)+\nabla\cdot\left(\sigma_h^n-\sigma^n\right),\mathbf{v}-\pi_h(\mathbf{v})\right)}{\|\ \nabla\mathbf{v}\ \|} + \frac{\left(\Delta t^{-1}\left(\mathbf{u}^n-\mathbf{u}_h^n\right)-\Delta t^{-1}\left(\mathbf{u}^{n-1}-\mathbf{u}_h^{n-1}\right),\mathbf{v}-\pi_h(\mathbf{v})\right)}{\|\ \nabla\mathbf{v}\ \|}\right]\right\}$$

where $\pi_h(\mathbf{v})$ is the L^2-orthogonal projection of $\mathbf{V}$ onto $\mathbf{V}_h$.

Then classical estimates (cf. [9]) and (11) imply that there exists another constant $\bar{C}(n)$, independent of h, such that for every n

$$\|\ p^n-p_h^n\ \| \leq \bar{C}(n)h\max_{0\leq i\leq n}\left\{\|\ \nabla\mathbf{u}^i\ \| + \|\ \nabla(\nabla\sigma^i)\ \| + \|\ \nabla(\nabla p^i)\ \|\right\}. \tag{12}$$

□

5 Stationary Case

To complete our analysis we briefly consider the approximation of stationary system (6) by means of the stationary counterpart of the finite element discretized problem (8), namely:

$$\left\{\begin{array}{l}\text{Find }\ (\bar{p}_h,\bar{\mathbf{u}}_h,\bar{\sigma}_h)\in Q_h\times\mathbf{V}_h^{\mathbf{g}}\times\Sigma_h \text{ such that}\\ \bar{a}\left((\bar{p}_h,\bar{\mathbf{u}}_h,\bar{\sigma}_h),(q,\mathbf{v},\tau)\right)=\bar{L}_h\left((q,\mathbf{v},\tau)\right)\ \ \forall\,(q,\mathbf{v},\tau)\in Q_h\times\mathbf{V}_h^0\times\Sigma_h\end{array}\right. \tag{13}$$

where for every $(p, \mathbf{u}, \sigma) \in Q \times \mathbf{V} \times \Sigma$ and $(q, \mathbf{v}, \tau) \in Q \times \mathbf{V} \times \Sigma$ we set

$$\begin{aligned} \bar{a}\left((p, \mathbf{u}, \sigma), (q, \mathbf{v}, \tau)\right) := \Delta t^2 \left(\nabla p - \nabla \cdot \sigma, \nabla q \right) + \Delta t \left(\mathbf{u}, \nabla \cdot \tau - \nabla q \right) + \\ \Delta t \left(\nabla p - \nabla \cdot \sigma, \mathbf{v} \right) + \frac{\Delta t}{2\eta} \left(\sigma, \tau \right) + \Delta t^2 \left(\nabla \cdot \sigma - \nabla p, \nabla \cdot \tau \right) \end{aligned} \tag{14}$$

and for given $\mathbf{f}$, $\mathbf{g}$, we set for every $(q, \mathbf{v}, \tau) \in Q \times \mathbf{V} \times \Sigma$

$$\bar{L}\left((q, \mathbf{v}, \tau)\right) = \Delta t^2 \left(\mathbf{f}, \nabla q - \nabla \cdot \tau \right) + \Delta t < \mathbf{g}, (\tau - Iq)\vec{\nu} >_{1/2, \partial\Omega} + \Delta t \left(\mathbf{f}, \mathbf{v} \right).$$

Proposition 5.1 *Problem (13) has a unique solution.*

Proof. First we note that problem (13) is equivalent to a linear system of algebric equations with an equal number of unknowns and equations. Therefore it has a unique solution if and only if it admits only the trivial solution, once its right hand side is set to zero.

Let us then assume that the triple $(\bar{p}_h, \bar{\mathbf{u}}_h, \bar{\sigma}_h)$ satisfies

$$\bar{a}\left((\bar{p}_h, \bar{\mathbf{u}}_h, \bar{\sigma}_h), (q, \mathbf{v}, \tau)\right) = 0 \quad \forall (q, \mathbf{v}, \tau) \in Q_h \times \mathbf{V}_h \times \Sigma_h$$

Taking $(q, \mathbf{v}, \tau) = (\bar{p}_h, \bar{\mathbf{u}}_h, \bar{\sigma}_h)$ we readily obtain

$$\Delta t^2 \parallel \nabla \cdot \bar{\sigma}_h - \nabla \bar{p}_h \parallel^2 + \frac{\Delta t}{2\eta} \parallel \bar{\sigma}_h \parallel = 0,$$

which implies that $\bar{\sigma}_h = 0$ and $\bar{p}_h = 0$.

This trivially implies that

$$(D(\bar{\mathbf{u}}_h), \tau - qI) = 0 \quad \forall (q, \tau) \in Q_h \times \Sigma_h \tag{15}$$

Next we endeavour to establish that relation (15) implies that $\bar{\mathbf{u}}_h = 0$. For this purpose we will take systematically $q = 0$. Now for every node P of T_h not belonging to $\partial\Omega$ we will choose N orthonormal frames B_P^i, $1 \leq i \leq N$, in such a way that one of the axes of B_P^i, say e_P^i, is the edge of an element of T_h having P as vertex. Now assume that P is the vertex of an element T_P such that $\bar{\mathbf{u}}_h$ vanishes at all the other N vertices of T_P. This is for instance the case of elements having an edge for $N = 2$ or a face for $N = 3$, contained in $\partial\Omega$. The axes e_P^i of B_P^i will be chosen in such a way that they are oriented from P to the vertices of T_P, say S_P^i for $1 \leq i \leq N$, respectively. Now we number the unit vector of B_P^i in such a way that e_P^i is the first one, and we take $\tau = \tau_P^i$ where τ_P^i is the tensor whose representation in terms of the frame B_P^i writes

$$\tau_P^i = \begin{bmatrix} f_P^i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where f_P^i is the function of S_h whose value equals one at S_P^i and zero at every other node of T_h. Finally defining the subset T_P^i of T_h as one of those elements having S_P^i as a common edge, by straightforward calculations we derive

$$\left(D(\bar{\mathbf{u}}_h), \tau_P^i\right) = \Sigma_{T \in T_P^i} \frac{area(T)}{N_T} \left(\frac{-\bar{\mathbf{u}}_h \cdot e_P^i}{l_P^i} \right)$$

where $l_P^i = \overline{PS_P^i}$. Letting i vary from one to N we immediately conclude from (15) that $\bar{\mathbf{u}}_h = \mathbf{0}$.

Now the question is: is it possible to find a path linking all the nodes of T_h, starting from a node P having N neighboring nodes on $\partial\Omega$, in such a way that every new node of the path has

N neighboring nodes at which it was previously established that $\bar{\mathbf{w}}_h$ vanishes. The answer is yes according to the following argument.

Once we eliminate from the mesh the set Γ_h^1 of all the elements of T_h having at least N vertices on $\partial\Omega$, in which $\mathbf{w}_h$ vanishes identically according to the above argument, we come up with a new domain $\Omega_h^1 \subset \Omega - \Gamma_h^1$. This is a set of elements of T_h in which $\mathbf{w}_h$ possibly does not vanish identically. If Ω_h^1 is empty the proof is complete. Otherwise $\mathbf{w}_h$ vanishes on the boundary of Ω_h^1, and this domain necessarily contains at least one element having exactly one vertex that does not belong to its boundary at which possibly $\mathbf{w}_h \neq \mathbf{0}$. More precisely such element has a common face with an element of $\Omega - \Omega_h^1$ and a vertex in the interior of Ω_h^1. Let Γ_h^2 be the union of all such elements. Then we apply the same construction for the elements of Γ_h^1, to those of Γ_h^2, thereby establishing that $\mathbf{w}_h$ vanishes identically in Γ_h^2 too. Again we come up with a sub domain $\Omega_h^2 \not\subseteq \Omega_h^1 \not\subseteq \Omega$, namely the union of all elements of Ω_h^1 in which possibly $\mathbf{w}_h$ does not vanish identically. If Ω_h^2 is empty the proof is complete. Otherwise the procedure continues in the same way until we reach a domain $\Omega_h^r \not\subseteq \Omega_h^{r-1} \cdots \not\subseteq \Omega_h^1 \not\subseteq \Omega$ for a certain integer r, which contains no element having more than one vertex that does not belong to its boundary at which possibly $\mathbf{w}_h \neq \mathbf{0}$. Finally treating all the elements of Ω_h^r in the same manner as those of Γ_h^1 we establish that $\mathbf{w}_h \equiv \mathbf{0}$ everywhere in Ω. □

Next we prove

Theorem 5.1 *For every $\Delta t > 0$ the solution $(p_h^n, \mathbf{u}_h^n, \sigma_h^n)$ of (8) converges to $(\bar{p}_h, \bar{\mathbf{u}}_h, \bar{\sigma}_h)$ in $L^2(\Omega) \times [L^2(\Omega)]^N \times [L^2(\Omega)]^{N\times N}$, as n goes to infinity.*

Proof. First we set $\bar{p}_h^n = p_h^n - \bar{p}_h$, $\bar{\mathbf{u}}_h^n = \mathbf{u}_h^n - \bar{\mathbf{u}}_h$, $\bar{\sigma}_h^n = \sigma_h^n - \bar{\sigma}_h$ and take $q = \bar{p}_h^n$, $\mathbf{v} = \bar{\mathbf{u}}_h^n$ and $\tau = \bar{\sigma}_h^n$ in both (8 and (13), thereby obtaining, after combining the resulting relations:

$$\begin{array}{r}\frac{2\eta}{\lambda+\Delta t} \parallel \bar{\mathbf{u}}_h^n \parallel^2 + \parallel \bar{\sigma}_h^n \parallel^2 - \frac{2\eta\Delta t}{\lambda+\Delta t}(\nabla\cdot\bar{\sigma}_h^n, \bar{\mathbf{u}}_h^n) + \frac{2\eta\Delta t}{\lambda+\Delta t}(\nabla\bar{p}_h^n, \bar{\mathbf{u}}_h^n) + \\ \frac{2\eta\Delta t^2}{\lambda+\Delta t} \parallel \nabla\cdot\bar{\sigma}_h^n - \nabla\bar{p}_h^n \parallel = \frac{2\eta}{\lambda+\Delta t}\left(\bar{\mathbf{u}}_h^{n-1}, \bar{\mathbf{u}}_h^n\right) + \frac{\lambda}{\lambda+\Delta t}\left(\bar{\sigma}_h^{n-1}, \bar{\sigma}_h^n\right) + \\ \frac{2\eta\Delta t}{\lambda+\Delta t}\left(\bar{\mathbf{u}}_h^{n-1}, \nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n\right)\end{array}$$

Setting $\alpha = \dfrac{\lambda}{\lambda+\Delta t}$ and $\beta = \dfrac{2\eta}{\lambda+\Delta t}$

$$\begin{array}{r}(1-\alpha) \parallel \bar{\sigma}_h^n \parallel^2 + \alpha \parallel \bar{\sigma}_h^n \parallel^2 + \beta\left[\parallel \bar{\mathbf{u}}_h^n \parallel^2 + \Delta t\,(\bar{\mathbf{u}}_h^n, \nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n] + \right. \\ \left.\Delta t^2 \parallel \nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n \parallel^2\right] \leq \beta\left(\bar{\mathbf{u}}_h^{n-1}, \bar{\mathbf{u}}_h^n + \Delta t\,(\nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n)\right) + \alpha\left(\bar{\sigma}_h^{n-1}, \bar{\sigma}_h^n\right)\end{array}$$

from which after simple calculations we derive: for all n,

$$\begin{array}{r}(1-\alpha) \parallel \bar{\sigma}_h^n \parallel^2 + \frac{\alpha}{2} \parallel \bar{\sigma}_h^n \parallel^2 + \frac{\beta}{2}\left[\parallel \bar{\mathbf{u}}_h^n \parallel^2 + \Delta t^2 \parallel \nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n \parallel^2\right] \\ \leq \frac{\beta}{2} \parallel \bar{\mathbf{u}}_h^{n-1} \parallel^2 + \frac{\alpha}{2} \parallel \bar{\sigma}_h^{n-1} \parallel^2\end{array}$$

This implies that $[\beta \parallel \bar{\mathbf{u}}_h^n \parallel^2 + \alpha \parallel \bar{\sigma}_h^n \parallel^2]/2$ is a decreasing sequence of positive numbers and hence a converging one. Therefore since $0 < \alpha < 1$ and $\beta > 0$ we have $\parallel \bar{\sigma}_h^n \parallel \to 0$ and $\parallel \nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n \parallel \to 0$. Recalling (7) the convergence of $\bar{\sigma}_h^n$ to zero in $[L^2(\Omega)]^{N\times N}$ implies that $\nabla\cdot\bar{\sigma}_h^n$ also converges to zero in $[L^2(\Omega)]^N$. It follows that $\parallel \nabla\bar{p}_h^n \parallel \to 0$ as n goes to ∞. On the other hand, according to [5], there exists a constant $C_B > 0$ such that $\parallel q \parallel \leq C_B \parallel \nabla q \parallel$ for every $q \in Q$. Hence $\bar{p}_h^n$ also tends to zero in $L^2(\Omega)$ as n goes to ∞.

As for the convergence of $\bar{\mathbf{u}}_h^n$ to zero, we employ an argument similar to the one of *Proposition 5.1*; indeed from the convergence to zero of $\bar{\sigma}_h^n$ and $\bar{p}_h^n$, we readily infer from (13) and (8) that

$$(\bar{\mathbf{u}}_h^n, \nabla \cdot \tau - \nabla q) \to 0 \quad \forall (q, \tau) \in (Q_h, \Sigma_h).$$

Then choosing $q = 0$ and $\tau = \tau_P^i$ (cf. *Proposition 5.1*), and sweeping the mesh in the way indicated in the proof of that result, we derive $\bar{\mathbf{u}}_h^n(P) \to 0$ for every vertex P of the mesh, and the result follows.

□

To conclude we prove the following convergence result

Theorem 5.2 *Assume that the solution of (5) belongs to* $H^2(\Omega) \times [H^1(\Omega)]^N \times [H^2(\Omega)]^{N \times N}$ *for every* n*. Then as* h *goes to zero the solution* $(\bar{p}_h, \bar{\mathbf{u}}_h, \bar{\sigma}_h)$ *of (13) converges to the solution* $(\bar{p}, \bar{\mathbf{u}}, \bar{\sigma})$ *of (6) in* $L^2(\Omega) \times [L^2(\Omega)]^N \times [L^2(\Omega)]^{N \times N}$*.*

Proof. Let $W = (\bar{p}, \bar{\mathbf{u}}, \bar{\sigma})$ and $W_h = (\bar{p}_h, \bar{\mathbf{u}}_h, \bar{\sigma}_h)$. We further set $W^n = (p^n, \mathbf{u}^n, \sigma^n)$ and $W_h^n = (p_h^n, \mathbf{u}_h^n, \sigma_h^n)$. Still denoting by $\| \cdot \|$ the norm of $L^2(\Omega) \times [L^2(\Omega)]^N \times [L^2(\Omega)]^{N \times N}$, we have:

$$\| W - W_h \| \leq \| W - W^n \| + \| W^n - W_h^n \| + \| W_h^n - W_h \|$$

Now for a given $\varepsilon > 0$ we choose n in such a way that $\| W - W^n \| < \varepsilon/3$ and $\| W_h^n - W_h \| < \varepsilon/3$, which is possible according to Theorems 3.1 and 5.1. Next for such n we choose h in such a way that $\| W^n - W_h^n \| < \varepsilon/3$, which is possible according to our regularity assumptions and to Theorem 4.1. This means that for every $\varepsilon > 0$ we may choose h in such a way that $\| W - W_h \| < \varepsilon$, and the result follows. □

6 A numerical example and final comments

In [7] the time dependent counterpart of the space-time discretization procedure adopted in this work was studied from both the theoretical and computational points of view. Here in order to illustrate what can be expected from this methodology in the solution of stationary problems we give a numerical example, in which the Stokes system (6) is solved, for a Poiseuille flow in a cylindrical duct with unit height, having a circular cross section with unit radius. Taking a unit viscosity and noticing that in this case $\mathbf{f} = \mathbf{0}$, the exact solution in terms of the variables (x, y, z) of a cartesian coordinate system, which the cylinder is referred to, is given by:

$$\mathbf{u}(x, y, z) = [0, 0, 1 - x^2 - y^2]^T$$

$$p(x, y, z) = 2 - 4z$$

$$\sigma(x, y, z) = \begin{bmatrix} 0 & 0 & -2x \\ 0 & 0 & -2y \\ -2x & -2y & 0 \end{bmatrix}.$$

In spite of the simplicity of this exact solution, the numerical solution can by no means be exact, due to the parabolic velocity profile.

Taking into account only the quarter domain given by $x > 0$ and $y > 0$ for symmetry reasons, we compute with spatial meshes generated in the following manner. First we construct a $L \times L \times L$ uniform mesh of a unit cube having the origin and the point $(1, 1, 1)$ as opposite vertices. Then each cubic element of this mesh is subdivided into six tetrahedra in the classical manner, using its diagonal parallel to the segment joining the origin to the point $(1, 1, 1)$. Next we map this mesh into a pseudo uniform mesh of the quarter cylinder by transforming the node cartesian coordinates (x, y) in each plane $z = m/L$, for $m = 0, \ldots, L$, into polar coordinates in the way described in [20]. Choosing $\lambda = 1$ and starting from the initial solution $\sigma^0 = \mathcal{O}$, together with $\mathbf{u}^0 = \mathbf{0}$ everywhere in the domain, except on the boundary portions given by $z = 0$ and $z = 1$ where $\mathbf{u}^0 = \mathbf{u}$, we compute up to the fictitious time $t = 5$, with a time step Δt equal to the inverse of $2,500\ L$. Both velocity and extra-stress are determined explicitly using the mass lumping technique [7], and as a matter of fact the code was run with only one iteration of the explicit solution algorithm per time step. This means that, except for the pressure computations, the solution method is fully explicit. Nevertheless for the pressure computations a standard preconditioned conjugate gradient algorithm was employed in connection with a fixed compact stored matrix resulting from the discretization of the underlying Poisson problem. Actually convergence of this algorithm was attained after only a few iterations at every time step.

Table 1: Relative errors in the L^2-norm for $\lambda = 1$, $\eta = 1$ and $\Delta t = L^{-1}/2,500$

L	$\mathbf{u}$	p	σ
2	0.7795×10^{-3}	$0.3445 \times 10^{+0}$	$0.2010 \times 10^{+0}$
4	0.9870×10^{-2}	$0.1438 \times 10^{+0}$	0.7676×10^{-1}
8	0.1121×10^{-1}	0.5431×10^{-1}	0.2954×10^{-1}
16	0.9977×10^{-2}	0.2177×10^{-1}	0.1272×10^{-1}

As one can infer from these results, the convergence of both p and σ as $h\tilde{-}1/L$ goes to zero appears to be better than linear. On the other hand the convergence rate of $\mathbf{u}$ is less clear, even if the velocity approximations are not so bad with a relative error around one percent. In this respect we should report that in some of our viscolelastic computations a convergent behavior of the velocity approximations was observed by pushing further the number of time steps (iterations) and applying a convergence criterion based on a given small tolerance, instead of keeping fixed a final fictitious time like in the above test. Some of these results were shown in [4], and new ones will be the object of a forthcoming paper on the full system governing stationary or time-dependent viscoelastic flow.

In [7] we proved that the method studied in this work is suitable for solving time-dependent viscoelastic flow problems. Referring also to [22] for its application to the solution of the stationary Navier-Stokes equations, as a conclusion we can say that the proposed method is also a reliable one to simulate a large spectrum of stationary incompressible flows expressed in terms of the three fields velocity, pressure and extra-stress tensor at a rather low cost. Indeed the solution procedure is practically explicit, thereby avoiding heavy storage requirements for a system, in which not less than ten strongly coupled unknown scalar functions are searched for in the three-dimensional case.

Acknowledgment

The authors wish to thank the anonymous referees for their careful reading of the manuscript and their fruitful comments and suggestions. The authors J. H. Carneiro de Araujo and V. Ruas gratefully acknowledge the financial support received from the agency CNPq through grants 304518/2002-6 and 307996/2008-5.

References

[1] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] A. Ainser, D. Dupuy, G. Panasenko and L. Sirakov. Flow in a Wavy Tube Structure: Asymptotic Analysis and Numerical Simulation. *Comptes Rendus de l'Académie des Sciences de Paris*, II-b(331-9):609–615, 2003

[3] J. Baranger and D. Sandri. Approximation par éléments finis d'écoulements de fluides viscoélastiques. Existence de solutions approchées et majorations d'erreur. Comptes Rendus de l'Académie des Sciences de Paris, I (312):541-544, 1991.

[4] A.P. Brasil Jr., J. H. Carneiro de Araujo and V. Ruas. A New Algorithm for Simulating Viscoelastic Flows Accommodating Piecewise Linear Finite Elements. Journal of Computational and Applied Mathemantics , 215:311-319, 2008.

[5] D. Braess. *Finite Elements* Cambridge University Press, Cambridge U.K., 1997.

[6] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods.* Springer-Verlag, Berlin, 1991.

[7] J.H. Carneiro de Araujo, P. Dias Gomes and V. Ruas. Study of a finite element method for the time-dependent generalized Stokes system associated with viscoelastic flow . Journal of Computational and Applied Mathematics, 234: 2562-2577, 2010.

[8] A. Chorin. Numerical simulation of the Navier-Stokes equations. *Mathematics of Computation*, 22:325-352, 1966.

[9] P. G. Ciarlet. *The Finite Element Method for Elliplic Problems.* North-Holland, Amsterdam, 1977.

[10] R. Codina and O.C. Zienkiewicz. CBS versus GLS stabilization of the incompressible Navier(Stokes equations and the role of the time step as a stabilization parameter *Communications in Numerical Methods in Engineering*, 18: 99-112, 2002.

[11] M. Fortin, R. Guénette and R. Pierre. Numerical Analysis of the Modified EVSS Method. *Computer Methods in Applied Mechanics and Engineering*, 143(1-2):79–95, 1997.

[12] L. Franca, T.J.R. Hughes, A.F.D.Loula and I.Miranda A new family of stable element for nearly incompressible elasticity based on a mixed Petrov-Galerkin finite element formulation. *Numerische Mathematik*, 53: 123-141, 1988.

[13] L. Franca and R. Stenberg. Error analysis of some Galerkin least-squares methods for the elasticity equations. *SIAM Journal of Numerical Analysis*, 26(6): 1680-1697, 1991.

[14] V. Girault and P. A. Raviart. *Finite Element Methods for Navier-Stokes Equations.* Springer-Verlag, Berlin, 1986.

[15] D. Goldberg and V. Ruas. A Numerical Study of Projection Algorithms in the Finite Element Simulation of Three-dimensional Viscous Incompressible Flow. *International Journal for Numerical Methods in Fluids*, 30: 233–256, 1999.

[16] J.L. Guermond and J. Shen. A new class of truly consistent spltting schemes for incompressible flows, *Journal of Computational Physics*. 192: 262-276, 2003.

[17] J. M. Marchal and M. Crochet. A New Mixed Finite Element for Calculating Viscoelastic Flow). *Journal of Non Newtonian Fluid Mechanics*, 26:77–117, 1987.

[18] H. Rahmoun and V. Ruas. Simulation numérique d'écoulements instationnaires de fluides incompressibles visqueux par la méthode des éléments finis appliquée dans divers domaines. Rapport de stage de DEA de mécanique, Université Pierre et Marie Curie, Paris, 1993.

[19] R. Rannacher. On Chorin's Projection Method for the Incompressible Navier-Stokes Equations. *Lecture Notes in Mathematics 1530, Springer Verlag, Berlin*, 1991.

[20] V. Ruas, Automatic generation of triangular finite element meshes, Comp. Maths. with Applications, 5: 125-140, 1979.

[21] V. Ruas, J. H. Carneiro de Araujo and M. A. M. Silva Ramos. Approximation of the Three-field Stokes System via Optimized Quadrilateral Finite Elements. *Mathematical Modelling and Numerical Analysis*, 27(1):107–127, 1993.

[22] V. Ruas and A.P. Brasil Jr. Explicit solution of the incompressible Navier-Stokes equations with linear finite elements. Applied Mathematics Letters , 20: 1005-1010, 2007.

[23] G. Strang and G. Fix. *An Analysis of the Finite Element Method*. Prentice Hall, Englewood Cliffs, 1973.

[24] R. Temam. Une méthode d'approximation de la solution des équations de Navier-Stokes. *Bulletin de la Société des Mathématiques de France*, 98: 115-152, 1968.