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Numerical Treatment of Singularly Perturbed Delay Differential Equations Using B-Spline Collocation Method on Shishkin Mesh[1]

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Abstract: This paper is devoted to the numerical study for a class of boundary value problems of second-order differential equations in which the highest order derivative is multiplied by a small parameter ϵ and both the convection and reaction terms are with negative shift. To obtain the parameter-uniform convergence, a piecewise uniform mesh (Shishkin mesh) is constructed, which is dense in the boundary layer region and coarse in the outer region. Parameter-uniform convergence analysis of the method has been given. The method is shown to have almost second-order parameter-uniform convergence. The effect of small delay δ on the boundary layer has also been discussed. To demonstrate the performance of the proposed scheme several examples having boundary layers have been carried out. The maximum absolute errors are presented in the tables.

Keywords: Singular perturbation, B-spline collocation method, Delay differential equations, Fitted mesh, Boundary layer

Mathematics Subject Classification: 34D15, 34E15, 34K10, 65L50

1 Introduction

A delay differential equation (DDE) is an equation where the evolution of the system at a certain time, depends on the state of the system at an earlier time. This is distinct from ordinary differential equations (ODEs) where the derivatives depend only on the current value of the independent

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variable. If we restrict the class of delay differential equations to a class in which the highest derivative is multiplied by a small parameter, then it is said to be a singularly perturbed delay differential equation. Delay differential equations arise in the mathematical modelling of various practical phenomena, for instance, micro scale heat transfer [32], hydrodynamics of liquid helium [13], second-sound theory [14], thermo-elasticity [7], diffusion in polymers [18], reaction-diffusion equations [2], stability [3], a variety of model for physiological processes or diseases [5, 19] etc.

A DDE is said to be of retarded delay differential equation (RDDE) if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral delay differential equation (NDDE). For example, the DDE

$$\begin{aligned}\phi^m(t) &= f(t, \phi(t), \phi'(t-\eta_1), \phi''(t-\eta_2), \ldots\ldots, \phi^{m-2}(t-\eta_{m-2}), \phi^{m-1}(t-\eta_{m-1})), \\ & t \in [t_0, t_0+T], \quad \eta_i \geq 0, \; \forall\, 1 \leq i \leq m-1, \\ & \phi(t_0) = \upsilon^0, \quad \phi^j(t) = \upsilon^j(t), \; j = 1(1)m-1, \quad t \in [t_0-\eta_i, t_0],\end{aligned}$$

is a retarded type of order m. On the other hand the DDE

$$\begin{aligned}\psi^k(t) &= g(t, \psi(t), \psi'(t-\gamma_1), \psi''(t-\gamma_2), \ldots\ldots, \psi^{k-1}(t-\gamma_{k-1}), \psi^k(t-\gamma_k)), \\ & t \in [t_0, t_0+T], \quad \gamma_i \geq 0, \; \forall\, 1 \leq i \leq k-1, \; \gamma_k \neq 0, \\ & \psi(t_0) = \nu^0, \quad \psi^j(t) = \nu^j(t), \; j = 1(1)k, \quad t \in [t_0-\gamma_i, t_0],\end{aligned}$$

is of neutral type of order k.

If we restrict it to a class in which the highest derivative term is multiplied by a small parameter, then we get singularly perturbed delay differential equations. This paper is devoted to second order singularly perturbed RDDEs. Frequently, DDEs have been reduced to ODEs with coefficients that depend on the delay by means of first-order accurate Taylor's series expansions of the terms that involve delay.

Asymptotic as well as numerical methods for DDEs are in many ways more challenging to implement than for ODEs because we have to use an appropriate approximation for the retarded arguments $y(x-\delta)$ and its derivatives and the algorithm has to take care of the jump discontinues due to the delay.

Parameter-uniform numerical methods [8, 21] are methods whose numerical approximations U^N satisfy error bounds of the form

$$\|u_\epsilon - U^N\| \leq C\vartheta(N), \quad \vartheta(N) \to 0 \text{ as } N \to \infty,$$

where u_ϵ is the solution of the continuous problem, $\|.\|$ is the maximum point-wise norm, N is the number of mesh points (independent of ϵ) used and C is positive constant which is independent of both ϵ and N. In other words, the numerical approximations U^N converge to u_ϵ for all values of ϵ in the range $0 < \epsilon \ll 1$.

It is well-known that standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ϵ is small. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy does not depend on the parameter value ϵ, i.e., methods that are convergent ϵ-uniformly [6, 9, 28]. In this paper, the strategy and the proposed method based on a suitably designed fitted mesh has shown to converge with $\vartheta(N) = N^{-2}\ln^3 N$.

2 Statement of the Problem

In this paper, we consider the boundary value problems for a class of singularly perturbed differential difference equations with delay term in both the convection and the reaction terms.

Consider the second-order singularly perturbed differential-difference equation on the domain $\Omega = (0, 1)$

$$Ly(x) \equiv \epsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x-\delta) + c(x)y(x) = f(x), \tag{1}$$

subject to the conditions

$$y(x) = \phi(x), \quad \text{on } -\delta \le x < 0, \qquad y(1) = \gamma, \tag{2}$$

where the functions $a(x)$, $b(x)$, $c(x)$, $f(x)$ and $\phi(x)$ are smooth; ϵ is a small parameter $0 < \epsilon \ll 1$, the delay parameter $\delta(\epsilon)(0 < \delta \ll 1)$ is of $o(\epsilon)$ satisfying the condition $\epsilon - \delta a(x) + (\delta^2/2)b(x) > 0$. It is also assumed that $b(x) + c(x) \le -\theta < 0$, $\forall x \in \bar{\Omega}$ for some positive constant θ. When δ is zero the above equation reduces to a singularly perturbed ordinary differential equation with one parameter problem and has been well studied numerically [6, 8, 9, 21, 28] and asymptotically [4, 22, 23, 25, 26, 33]. It is well-known that for $\delta = 0$ problem (1) and (2) with small ϵ exhibits layers and turning points depending upon the coefficient of the convection and the reaction terms respectively, if $a(x)$ is positive through out the interval $[0, 1]$, then the boundary layer exists on the left side of the interval $[0, 1]$ and if $a(x)$ is negative throughout the interval $[0, 1]$, then the boundary layer exists on the right side of the interval $[0, 1]$. The zeros of $a(x)$ are called turning points. For the detailed theory of DDEs, readers can refer to [11, 24, 29].

In this paper, we present Taylor's approximation for the retarded terms, namely first-order approximation to the convection retarded term and second-order approximation to the reaction retarded term. Then the resulting differential-difference equation is solved by B-spline collocation method on Shishkin mesh.

We use the following Taylor's series to approximate the retarded argument

$$y(x-\delta) \approx y(x) - \delta y'(x) + (\delta^2/2)y''(x), \tag{3}$$

and

$$y'(x-\delta) \approx y'(x) - \delta y''(x). \tag{4}$$

Using (3) and (4) in the Eqs. (1) and (2), we get the approximating equation

$$(\epsilon - \delta a(x) + (\delta^2/2)b(x))y''(x) + (a(x) - \delta b(x))y'(x) + (b(x) + c(x))y(x) = f(x), \tag{5}$$

with

$$y(0) = \phi_0 = \phi(0), \qquad y(1) = \gamma. \tag{6}$$

Here, we assume that $a(x) \ge M > 0$, $\epsilon - \delta a(x) + (\delta^2/2)b(x) > 0$ and $a(x) - \delta b(x) \ge \beta > 0$ for all $x \in \bar{\Omega}$, where β is some positive constant. In this case the solution of the problem exhibits layer behaviour on the left side of the domain. The other case $a(x) - \delta b(x) \le -\beta < 0$ for all $x \in \bar{\Omega}$, implies the occurrence of boundary layer near right side of the domain and can be treated similarly.

3 Mesh Selection Strategy

It is well-known that on an equidistant mesh no scheme can attain convergence at all mesh points uniformly in ϵ, unless its coefficients have an exponential property. Therefore, unless we use a specially chosen mesh we will not be able to get ϵ uniform convergence at all the mesh points [28]. If a priori information is available about the exact solution then we could go for a highly non-uniform grid like [1, 10, 35]. The simplest possible non-uniform mesh, namely a piecewise-uniform mesh proposed by Shishkin [31], is sufficient for the construction of an ϵ-uniform method. Shishkin mesh is fine near layers but coarser otherwise. It is attractive because of its simplicity and adequate for handling a wide variety of singularly perturbed problems and in our case mostly the exact solution of delay differential equations is not known, hence a priori information about the solution is far from imagination. In this section, we construct the piecewise uniform mesh in such a way that more mesh points are generated in the boundary layer regions than outside these regions.

In the case of left boundary layer, we divide the given interval $[0,1]$ into two subintervals $[0,\tau]$ and $[\tau,1]$, where the transition parameter τ is given by $\tau = \min(1/2, K(c_\epsilon/\beta)\ln N)$, where K is some positive constant which depends on the scheme being used, $c_\epsilon = (\epsilon - \delta M + \delta^2 M')$, where $2M'$ is an upper bound of $b(x)$ and N is the number of discretization points. It is assumed that $N = 2^r$, where $r \geq 2$ is an integer, which guarantees that there will lie at least one point in the boundary layer. Then we have

$$x_i = \begin{cases} ih_i, & \text{if} \quad i = 0,1,\ldots,N/2, \\ \tau + (i - N/2)h_i, & \text{if} \quad i = N/2+1,\ldots,N, \end{cases}$$

where

$$h_i = \begin{cases} 2\tau/N, & \text{if} \quad i = 1,2,\ldots,N/2, \\ 2(1-\tau)/N, & \text{if} \quad i = N/2+1, N/2+2,\ldots,N, \end{cases}$$

i.e., the piecewise uniform mesh spacing h is $2\tau/N$ for the interval $[0,\tau]$ and $2(1-\tau)/N$ for the interval $[\tau,1]$.

Similarly if boundary layer exists at $x = 1$, then we define

$$x_i = \begin{cases} ih_i, & \text{if} \quad i = 0,1,\ldots,N/2, \\ 1 - \tau + (i - N/2)h_i, & \text{if} \quad i = N/2+1,\ldots,N, \end{cases}$$

where

$$h_i = \begin{cases} 2(1-\tau)/N, & \text{if} \quad i = 1,2,\ldots,N/2, \\ 2\tau/N, & \text{if} \quad i = N/2+1, N/2+2,\ldots,N, \end{cases}$$

i.e., the piecewise uniform mesh spacing h is $2(1-\tau)/N$ for the interval $[0, 1-\tau]$ and $2\tau/N$ for the interval $[1-\tau, 1]$.

Let $\bar{\Omega}_N = \{x_i\}_{i=0}^N$ be the set of mesh points. If the transition point τ is chosen independently of N, then one cannot obtain a convergence result that is uniform in ϵ [20].

4 B-Spline Collocation Method

In this section, we describe the B-spline collocation method to obtain the approximate solution of boundary value problem (5)–(6). Let $\pi \equiv \{0 = x_0 < x_1 < x_2 \ldots < x_{N-1} < x_N = 1\}$, be the partition of $\bar{\Omega}$, where $h = x_{i+1} - x_i,\ i = 0,1\ldots,N-1$.

We have used cubic B-splines to approximate the solution $y(x)$ of the equation (5). The cubic B-splines $B_i,\ i = -1,0,\ldots N+1$ at the nodes x_i are defined to form a basis over the interval

$[0,1]$ (see ref. [27]). Thus an approximation $S(x)$ to the exact solution $y(x)$ of the boundary value problem (5)–(6), can be expressed in terms of B-splines as

$$S(x) = \sum_{i=-1}^{N+1} \alpha_i B_i(x), \tag{7}$$

where α_i are unknown real coefficients to be determined by requiring that $S(x)$ satisfies (5) at $N+1$ collocation points and boundary conditions.

A cubic B-splines $B_i,\ i = -1, 0, \ldots, N+1$, covers four elements and defined over the interval $[0,1]$ as follows:

$$B_i(x) = \frac{1}{h^3}\begin{cases} (x - x_{i-2})^3, & x_{i-2} \le x \le x_{i-1}, \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x_{i-1} \le x \le x_i, \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x_i \le x \le x_{i+1}, \\ (x_{i+2} - x)^3, & x_{i+1} \le x \le x_{i+2}, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

It is easy to see that each $B_i(x)$ is also a piecewise cubic with knots at $\bar{\Omega}_N$, also $B_i(x)$ is twice continuously differentiable $\forall x \in \mathbb{R}$. Also the values of $B_i(x)$, $B_i'(x)$ and $B_i''(x)$ at the nodal points x_i's are given as in table given below:

Table 1: B-Spline basis values

	Nodal values				
	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B_i'(x)$	0	$3/h$	0	$-3/h$	0
$B_i''(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

So the four cubic B-splines $B_{i-1}, B_i, B_{i+1}, B_{i+2}$ lie in each element. It can be easily seen that over the typical element $[x_i, x_{i+1}]$, the approximate solution $S(x)$ is given by

$$S(x) = \sum_{j=i-1}^{i+2} \alpha_j B_j(x). \tag{9}$$

The form (9) shows the variation of all contributing cubic B-splines over a single element and is useful for working out the solution inside the element. At nodal points values of S and its derivatives S' and S'' can be determined in terms of the element parameters α_i:

$$\left.\begin{aligned} S(x_i) &= \alpha_{i-1} + 4\alpha_i + \alpha_{i+1} \\ hS'(x_i) &= 3(\alpha_{i+1} - \alpha_{i-1}) \\ h^2 S''(x_i) &= 6(\alpha_{i-1} - 2\alpha_i + \alpha_{i+1}) \end{aligned}\right\}, \tag{10}$$

where $'$ and $''$ denotes the first and second differentiations with respect to x, respectively.

Let $S_3(\pi)$ be the set of all cubic spline functions over the partition π. Let $\mathbf{B} = \{B_{-1}, B_0, B_1, \ldots, B_{N+1}\}$ and let $\Phi_3(\bar{\Omega}_N)$ be the set of all linear combinations of B_i's. The functions B_i's are linearly independent, thus $\Phi_3(\bar{\Omega}_N)$ is $(N+3)$-dimensional subspace of $S_3(\pi)$(see ref. [30]).

To apply the collocation method, collocation points are selected to coincide with nodes and then substituting the values of S_i and first two successive derivatives S_i' and S_i'' at nodal points into equation (5). This yields a system of $(N+1)$ linear equations in $(N+3)$ unknowns.

$$r_i^- \alpha_{i-1} + r_i^c \alpha_i + r_i^+ \alpha_{i+1} = h^2 f_i, \quad 0 \le i \le N, \tag{11}$$

where

$$\begin{aligned} r_i^- &= 6(\epsilon - \delta a_i + (\delta^2/2) b_i) - 3(a_i - \delta b_i)h + (b_i + c_i)h^2, \\ r_i^c &= -12(\epsilon - \delta a_i + (\delta^2/2) b_i) + 4(b_i + c_i)h^2, \\ r_i^+ &= 6(\epsilon - \delta a_i + (\delta^2/2) b_i) + 3(a_i - \delta b_i)h + (b_i + c_i)h^2, \end{aligned}$$

where $a(x_i) = a_i$, $b(x_i) = b_i$, $c(x_i) = c_i$ and $f(x_i) = f_i$. The given boundary conditions become

$$\alpha_{-1} + 4\alpha_0 + \alpha_1 = \phi_0, \tag{12a}$$
$$\alpha_{N-1} + 4\alpha_N + \alpha_{N+1} = \gamma. \tag{12b}$$

Thus the Eqs. (11) and (12) lead to a $(N+3) \times (N+3)$ system with $(N+3)$ unknowns $\alpha = \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_{N+1}$. Now eliminating α_{-1} from first equation of (11) and (12a), we get

$$(r_0^c - 4r_0^-)\alpha_0 + (r_0^+ - r_0^-)\alpha_1 = h^2 f_0 - r_0^- \phi_0. \tag{13}$$

Again, eliminating α_{N+1} from the last equation of (11) and from (12b), we obtain

$$(r_N^- - r_N^+)\alpha_{N-1} + (r_N^c - 4r_N^+)\alpha_N = h^2 f_N - \gamma r_N^+. \tag{14}$$

Now taking equations (13) and (14) with the second through (N-1)st equations of (11) we lead to a system of $(N+1)$ linear equations

$$A\alpha = d, \tag{15}$$

in $(N+1)$ unknowns $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{N-1}, \alpha_N$ with right hand side $d = (d_0, d_1, \ldots, d_{N-1}, d_N)^t$ and the co-efficient matrix A is given by

$$\begin{pmatrix} (r_0^c - 4r_0^-) & (r_0^+ - r_0^-) & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ r_1^- & r_1^c & r_1^+ & 0 & \ldots & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & r_i^- & r_i^c & r_i^+ & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & r_{N-1}^- & r_{N-1}^c & r_{N-1}^+ \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & (r_N^- - r_N^+) & (r_N^c - 4r_N^+) \end{pmatrix}.$$

The elements of the column vector d are

$$d_j = \begin{cases} h^2 f_0 - r_0^- \phi_0, & j = 0, \\ h^2 f_j, & j = 1(1)N-1, \\ h^2 f_N - \gamma r_N^+, & j = N. \end{cases}$$

It can be easily seen that the matrix A is strictly diagonally dominant and hence nonsingular. Since A is nonsingular, we can solve the system $A\alpha = d$ for $\alpha_0, \alpha_1, \ldots, \alpha_N$ and substitute into the boundary condition (12) to obtain α_{-1} and α_{N+1}. Hence the method of collocation using a basis of cubic B-splines applied to problem (5)-(6) has a unique solution $S(x)$ given by (7).

5 Uniform Convergence

In this section, we shall show that the B-spline collocation method described in previous section is parameter-uniform convergent of order almost two. Note that throughout the proof we take C as a positive generic constant independent of ϵ, δ and the discretization parameter N, which can take different values at different places. We define $\bar{h} = \max_{1\le i\le N}\{x_i - x_{i-1}\} = \max h_i$ and $\underline{h} = \min_{1\le i\le N}\{x_i - x_{i-1}\} = \min h_i$, we also define $\beta' = \bar{h}/\underline{h}$. For the derivation of uniform convergence, we use the following lemmas:

Lemma 5.1. *[27] The B-splines $B_{-1}, B_0, \ldots, B_{N+1}$, satisfy the inequality*

$$\sum_{i=-1}^{N+1} | B_i(x) | \le 10, \qquad 0 \le x \le 1.$$

Proof. For the proof of the lemma readers can refer to [15]. □

Lemma 5.2. *(Kellogg and Tsan [17]) If $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth and independent of ϵ, then the solution $y(x)$ of (1) and (2) satisfies*

$$| y^{(k)}(x) | \le C(1 + c_\epsilon^{-k} e^{-\beta x/c_\epsilon}), \quad k = 0, 1, 2 \ldots$$

Proof. For the proof of the lemma readers can refer to [21]. □

Theorem 5.1. *Let $S(x)$ be the collocation approximation from the space of cubic splines $\phi_3(\bar{\Omega}_N)$ to the solution $y(x)$ of the boundary value problem (1) and (2). If $f \in C^2[0,1]$, then the parameter-uniform error estimate is given by*

$$\sup_{0<\epsilon\le 1} \max_{0\le i\le N} | y(x_i) - S(x_i) | \le CN^{-2} \ln^3 N,$$

where C is a positive constant independent of ϵ, δ and N.

Proof. To estimate the error $| y(x) - S(x) |$, let $Y(x)$ be the unique spline interpolate from $\Phi_3(\bar{\Omega}_N)$ to the solution $y(x)$ of our boundary value problem (1) and (2) given by

$$Y(x) = \sum_{i=-1}^{N+1} \tilde{\alpha}_i B_i(x). \tag{16}$$

If $f(x) \in C^2[0,1]$ then $y(x) \in C^4[0,1]$ and it follows from Hall error estimates [12] that

$$\|D^j(y - Y)\| \le \lambda_j \|y^{(4)}\| \bar{h}^{4-j}, \quad j = 0, 1, 2, 3, \tag{17}$$

where λ_j are constants given by $\lambda_0 = 5/384,\ \lambda_1 = (1/216)(9+\sqrt{3}),\ \lambda_2 = (1/12)(3\beta'+1),\ \lambda_3 = (1/2)(\beta'^2+1)$. It follows immediately from the estimates (17) that

$$\begin{aligned} \mid Ly(x_i) - LY(x_i) \mid &= \mid \epsilon - \delta a(x_i) + (\delta^2/2)b(x_i) \mid\mid y''(x_i) - Y''(x_i) \mid \\ &+ \mid a(x_i) - \delta b(x_i) \mid\mid y'(x_i) - Y'(x_i) \mid \\ &+ \mid b(x_i) + c(x_i) \mid\mid y(x_i) - Y(x_i) \mid \\ &\leq (c_\epsilon \lambda_2 \bar{h}^2 + (\|a\| + \delta\|b\|)\lambda_1 \bar{h}^3 + \theta\lambda_0 \bar{h}^4)\|y^{(4)}\|. \end{aligned} \tag{18}$$

Using Lemma 5.2, we obtain

$$\mid Ly(x_i) - LY(x_i) \mid \;\leq\; C(c_\epsilon \lambda_2 \bar{h}^2 + (\|a\| + \delta\|b\|)\lambda_1 \bar{h}^3 + \theta\lambda_0 \bar{h}^4)(1 + c_\epsilon^{-4} e^{-\beta x/c_\epsilon}). \tag{19}$$

Now there arise two cases:

Case (i) $\tau = 1/2$. In this case, we have $K(c_\epsilon/\beta)\ln N \geq 1/2$ this gives $c_\epsilon^{-1} \leq C \ln N$ also $\bar{h} = 1/N$, therefore from (19) we get

$$\mid Ly(x_i) - LY(x_i) \mid \;\leq\; CN^{-2}\ln^3 N. \tag{20}$$

Case (ii) $\tau = K(c_\epsilon/\beta)\ln N$. In this case, first suppose i satisfies $N/2 \leq i \leq N$. Then it is easy to know from Lemma 5.2 that in the regular region i.e., $(\tau, 1)$

$$\mid y^{(k)}(x) \mid \leq C,$$

therefore we have immediately from (18)

$$\mid Ly(x_i) - LY(x_i) \mid \;\leq\; CN^{-2}. \tag{21}$$

On the other hand if i satisfies $1 \leq i \leq N/2$, in this case we have $\bar{h} = 2\tau/N = 2KN^{-1}(c_\epsilon/\beta)\ln N$, this gives $\bar{h}/c_\epsilon = CN^{-1}\ln N$, Then using the Lemma 5.2 given in [16], we get from (19)

$$\mid Ly(x_i) - LY(x_i) \mid \;\leq\; CN^{-2}\ln^2 N. \tag{22}$$

Combining (21) and (22) we get

$$\mid Ly(x_i) - LY(x_i) \mid \;\leq\; CN^{-2}\ln^2 N. \tag{23}$$

Finally on combining (20) and (23) we obtain

$$\mid Ly(x_i) - LY(x_i) \mid \;\leq\; CN^{-2}\ln^3 N. \tag{24}$$

Thus we have

$$\mid LS(x_i) - LY(x_i) \mid = \mid f(x_i) - LY(x_i) \mid = \mid Ly(x_i) - LY(x_i) \mid \leq CN^{-2}\ln^3 N. \tag{25}$$

Now suppose that $LY(x_i) = \tilde{f}(x_i)\ \forall 0 \leq i \leq N$ with the boundary conditions

$$Y(x_0) = \phi_0, \quad Y(x_N) = \gamma,$$

leads to the linear system $A\tilde{\alpha} = \tilde{d}$, then it follows that

$$A(\alpha - \tilde{\alpha}) = (d - \tilde{d}), \tag{26}$$

where

$$\begin{aligned}\alpha - \tilde{\alpha} &= (\alpha_0 - \tilde{\alpha}_0, \alpha_1 - \tilde{\alpha}_1, \ldots, \alpha_N - \tilde{\alpha}_N)^t,\\ d - \tilde{d} &= (\bar{h}^2(f(x_0) - \tilde{f}(x_0)), \bar{h}^2(f(x_1) - \tilde{f}(x_1)), \ldots, \bar{h}^2(f(x_N) - \tilde{f}(x_N)))^t.\end{aligned}$$

Using (25) we obtain

$$\|d - \tilde{d}\| \leq CN^{-4} \ln^3 N. \tag{27}$$

We have seen that, for sufficiently small values of $\bar{h}$, the coefficient matrix A is strictly diagonally dominant. Also note that, except the first row and the last row of A, the off diagonal elements $a_{i,i-1},\ a_{i,i+1},\ 1 \leq i \leq N-1$, are positive and the main diagonal elements $a_{i,i}$ are negative, therefore

$$| a_{i,i} | -(| a_{i,i-1} | + | a_{i,i+1} |) = -6(b_i + c_i)\bar{h}^2 \geq 6\theta\bar{h}^2 > 0, \text{ as } b(x) + c(x) \leq -\theta < 0.$$

Also from the first row of A, we have

$$| a_{0,0} | - | a_{0,1} | = 36(\epsilon - \delta a_0 + (\delta^2/2)b_0) - 6(a_0 - \delta b_0)\bar{h}.$$

Again from last row of A, we have

$$| a_{N,N} | - | a_{N,N-1} | = 36(\epsilon - \delta a_N + (\delta^2/2)b_N) + 6(a_N - \delta b_N)\bar{h}.$$

Therefore, by the estimate given in [34], we get

$$\|A^{-1}\| \leq CN^2. \tag{28}$$

From (26), (27) and (28), we get

$$| \alpha_i - \tilde{\alpha}_i | \leq CN^{-2} \ln^3 N, \quad 0 \leq i \leq N.$$

Now to estimate $| \alpha_{-1} - \tilde{\alpha}_{-1} |$ and $| \alpha_{N+1} - \tilde{\alpha}_{N+1} |$, using the boundary condition (12a), we obtain

$$| \alpha_{-1} - \tilde{\alpha}_{-1} | \leq CN^{-2} \ln^3 N.$$

Again using the boundary condition (12b), we obtain

$$| \alpha_{N+1} - \tilde{\alpha}_{N+1} | \leq CN^{-2} \ln^3 N.$$

Therefore

$$\max_{-1 \leq i \leq N+1} | \alpha_i - \tilde{\alpha}_i | \leq CN^{-2} \ln^3 N. \tag{29}$$

The above inequality enables us to estimate $| S(x) - Y(x) |$, we have

$$S(x) - Y(x) = \sum_{i=-1}^{N+1} (\alpha_i - \tilde{\alpha}_i) B_i(x).$$

Thus, using (29) and Lemma 5.1, we obtain

$$\begin{aligned}| S(x) - Y(x) | &\leq \max_{-1 \leq i \leq N+1} | \alpha_i - \tilde{\alpha}_i | \sum_{i=-1}^{N+1} | B_i(x) |,\\ &\leq CN^{-2} \ln^3 N.\end{aligned}$$

This gives

$$\max_{0\le i\le N} \mid S(x_i) - Y(x_i) \mid \le CN^{-2}\ln^3 N.$$

Therefore using triangle inequality, we obtain

$$\sup_{0<\epsilon\le 1} \max_{0\le i\le N} \mid y(x_i) - S(x_i) \mid \le CN^{-2}\ln^3 N.$$

Hence the result. □

6 Numerical Results and Discussions

To demonstrate the efficiency of the method, we solved several examples having boundary layers. Since the exact solution of the considered problems are not known so the maximum absolute errors are estimated by using the double mesh principle [6] and maximum nodal error is defined by

$$E_\epsilon^N = \max_{0\le i\le N} \mid y_i^N - y_{2i}^{2N} \mid,$$

where y_{2i}^{2N} is the solution obtained on a mesh containing the same number N of Shishkin mesh used to compute y_i^N and N more mesh points are added by selecting the mid points of all $\{x_i\}'$s, i.e., $x_{i+1/2} = (x_i + x_{i+1})/2$, for $i = 1, 2, \ldots, N-1$. The ϵ-uniform maximum nodal error is defined by

$$E^N = \max_{0<\epsilon\ll 1} E_\epsilon^N.$$

The numerical rates of convergence are defined as in [6]:

$$r_\epsilon^N = \ln_2(E_\epsilon^N / E_\epsilon^{2N}),$$

and the numerical rates of ϵ-uniform convergence are computed using

$$r^N = \ln_2(E^N / E^{2N}).$$

We also compute the constant in the error estimate, i.e., since we have the theoretical error bound $E^N \le CN^{-2}\ln^3 N$ from Theorem 5.1 we approximate the constant in the error estimate by $C^N = E^N N^2 / \ln^3 N$. Table 5 display the values of C^N for the examples taken.

Example 6.1. *First consider the example*

$$\epsilon y''(x) + (1+x)y'(x-\delta) + \exp(-2x)y(x-\delta) - 2\exp(-x)y(x) = 0,$$

under the interval and boundary conditions

$$y(x) = 1, \ for \ -\delta \le x < 0, \quad y(1) = 0.$$

Example 6.2. *Now we consider the example*

$$\epsilon y''(x) - (1+x)y'(x-\delta) + \exp(-2x)y(x-\delta) - \exp(-x)y(x) = 0,$$

under the interval and boundary conditions

$$y(x) = 1, \ for \ -\delta \le x < 0, \quad y(1) = -1.$$

Table 2: Maximum absolute error for example 6.1 with uniform mesh

ϵ	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{0}	2.5934E-04	6.5141E-05	1.6344E-05	4.0871E-06	1.0218E-06	2.5547E-07
2^{-4}	7.9711E-02	2.2731E-02	5.2094E-03	1.2763E-03	3.1748E-04	7.9272E-05
2^{-8}	2.0097E+00	1.1967E+00	7.4461E-01	3.6636E-01	1.0877E-01	2.6403E-02
2^{-12}	3.4110E+00	3.0794E+00	2.3668E+00	1.7568E+00	1.4875E+00	1.1389E+00
2^{-16}	3.5764E+00	3.5799E+00	3.4906E+00	3.1629E+00	2.4865E+00	1.9646E+00
2^{-20}	3.5874E+00	3.6189E+00	3.6296E+00	3.6112E+00	3.5129E+00	3.1855E+00
2^{-24}	3.5881E+00	3.6214E+00	3.6389E+00	3.6463E+00	3.6441E+00	3.6194E+00
2^{-28}	3.5881E+00	3.6216E+00	3.6394E+00	3.6486E+00	3.6528E+00	3.6535E+00
2^{-32}	3.5881E+00	3.6216E+00	3.6395E+00	3.6487E+00	3.6534E+00	3.6556E+00

Table 3: Maximum absolute error for example 6.1 with Shishkin mesh

ϵ	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$	$N=2048$
2^{0}	2.5934E-04	6.5141E-05	1.6344E-05	4.0871E-06	1.0218E-06	2.5547E-07	6.3868E-08	1.5968E-08
2^{-4}	1.0876E-02	4.2631E-03	1.5109E-03	5.1353E-04	1.6752E-04	5.2959E-05	1.6347E-05	4.9446E-06
2^{-8}	1.1150E-02	4.3896E-03	1.5670E-03	5.2887E-04	1.7240E-04	5.4529E-05	1.6827E-05	5.0895E-06
2^{-12}	1.1177E-02	4.3958E-03	1.5692E-03	5.2997E-04	1.7278E-04	5.4634E-05	1.6859E-05	5.0990E-06
2^{-16}	1.1179E-02	4.3971E-03	1.5697E-03	5.2999E-04	1.7277E-04	5.4637E-05	1.6861E-05	5.0996E-06
2^{-20}	1.1180E-02	4.3972E-03	1.5697E-03	5.3001E-04	1.7278E-04	5.4637E-05	1.6861E-05	5.0996E-06
2^{-24}	1.1180E-02	4.3972E-03	1.5697E-03	5.3002E-04	1.7278E-04	5.4637E-05	1.6861E-05	5.0995E-06
2^{-28}	1.1180E-02	4.3972E-03	1.5697E-03	5.3002E-04	1.7278E-04	5.4637E-05	1.6861E-05	5.0995E-06
2^{-32}	1.1180E-02	4.3972E-03	1.5697E-03	5.3002E-04	1.7278E-04	5.4637E-05	1.6861E-05	5.0995E-06
E^{N}	1.1180E-02	4.3972E-03	1.5697E-03	5.3002E-04	1.7278E-04	5.4637E-05	1.6861E-05	5.0996E-06
r^{N}	1.3463	1.4861	1.5664	1.6171	1.6610	1.6962	1.7252	—

The numerical results presented in Table 2 clearly indicate that the proposed scheme with uniform mesh is not uniformly convergent for sufficiently small value of ϵ and the maximal nodal error increases as ϵ decreases. To overcome this drawback, we have used a class of special piecewise-uniform meshes known as Shishkin meshes. The numerical results displayed in Tables 3 and 4 clearly indicate that the proposed method based on a B-spline collocation with Shiskin mesh is ϵ-uniformly convergent. The proposed numerical method is accurate of order almost two.

For examples 6.1 and 6.2 we have plotted the graphs of the computed solutions for $\epsilon = 10^{-1}$ and $\epsilon = 10^{-2}$ and various values of δ, as shown in figures 1-4. For all figures we use $N = 100$. It was observed that when δ increases the width of the boundary layer increases in the case of right boundary layer problems and in the case of left boundary layer problems the width of the boundary layer decreases when delta increases. This shows clearly the effect of δ on the boundary layers.

Since the exact solution is unknown, for problem 6.1 we plot the graphs taking two values of N, 100 and 200, for fixed value of $\delta = 0.5 \times \epsilon$ and for $\epsilon = 0.01$ with uniform mesh (see Fig. 5) and with Shishkin mesh (see Fig. 6). From figures it can be easily seen that how much better approximation the present method gives on Shishkin mesh than uniform mesh. This show the

Table 4: Maximum absolute error for example 6.2 with Shishkin mesh

ϵ	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{0}	1.1305E-04	2.8235E-05	7.0585E-06	1.7647E-06	4.4117E-07	1.1029E-07	2.7573E-08
2^{-4}	5.1826E-02	1.1491E-02	2.7950E-03	6.9407E-04	1.7323E-04	4.3289E-05	1.0821E-05
2^{-8}	1.0445E-01	3.7010E-02	1.2611E-02	4.2314E-03	1.4187E-03	4.6812E-04	1.5501E-04
2^{-12}	1.0383E-01	3.6669E-02	1.2541E-02	4.2449E-03	1.4298E-03	4.7508E-04	1.5924E-04
2^{-16}	1.0389E-01	3.6701E-02	1.2463E-02	4.1666E-03	1.4086E-03	4.7513E-04	1.5946E-04
2^{-20}	1.0389E-01	3.6709E-02	1.2478E-02	4.1763E-03	1.3898E-03	4.5369E-04	1.5379E-04
2^{-24}	1.0389E-01	3.6710E-02	1.2479E-02	4.1785E-03	1.3936E-03	4.5637E-04	1.4895E-04
2^{-28}	1.0389E-01	3.6710E-02	1.2479E-02	4.1786E-03	1.3939E-03	4.5697E-04	1.4992E-04
2^{-32}	1.389E-01	3.6710E-02	1.2479E-02	4.1786E-03	1.3939E-03	4.5701E-04	1.5000E-04
E^N	1.0445E-01	3.7010E-02	1.2611E-02	4.2449E-03	1.4298E-03	4.7513E-04	1.5946E-04
r^N	1.4968	1.5532	1.5709	1.5699	1.5894	1.5751	—

Table 5: E^N, r^N and C^N for examples 6.1 and 6.2

N	Example 6.1			Example 6.2		
	E^N	r^N	C^N	E^N	r^N	C^N
16	1.1180E-02	1.35	0.13	1.0445E-01	1.50	1.25
32	4.3972E-03	1.49	0.11	3.7010E-02	1.55	0.91
64	1.5697E-03	1.57	0.09	1.2611E-02	1.57	0.72
128	5.3002E-04	1.62	0.08	4.2449E-03	1.57	0.61
256	1.7278E-04	1.66	0.07	1.4298E-03	1.59	0.55
512	5.4637E-05	1.70	0.06	4.7513E-04	1.58	0.51
1024	1.6861E-05	1.73	0.05	1.5946E-04	1.53	0.50
2048	5.0996E-06	1.73	0.05	5.5076E-05	1.53	0.52

significance of Shishkin mesh over uniform mesh.

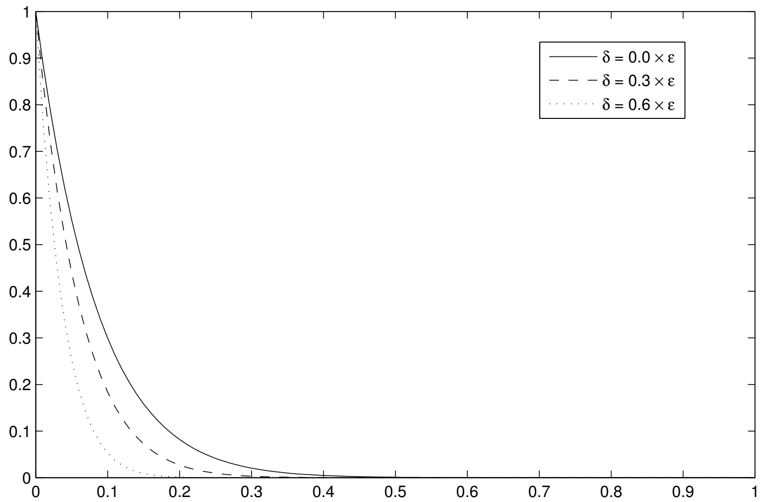

Figure 1: Approximate solutions for problem 6.1 for $\epsilon = 0.1$ and different values of δ

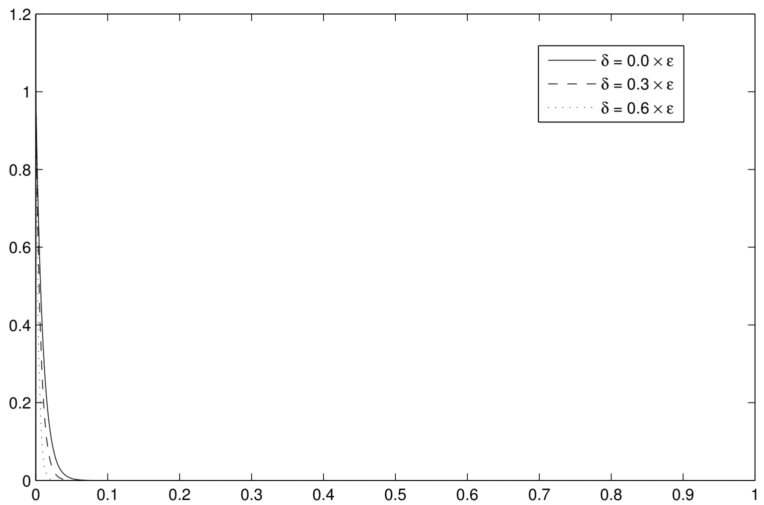

Figure 2: Approximate solutions for problem 6.1 for $\epsilon = 0.01$ and different values of δ.

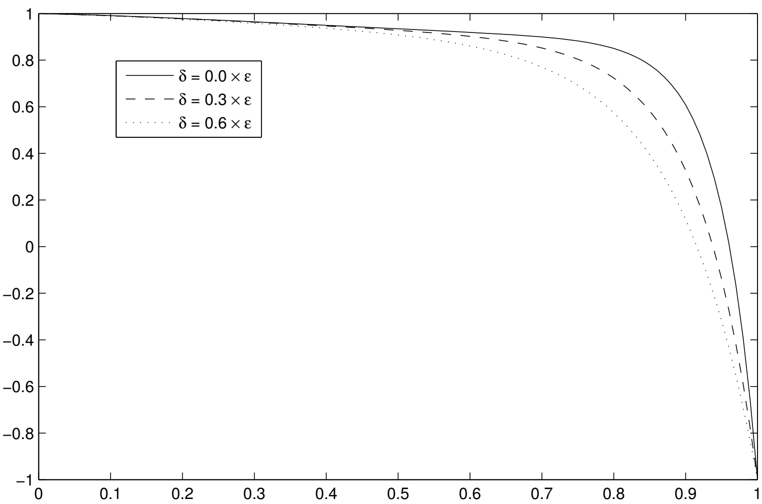

Figure 3: Approximate solutions for problem 6.2 for $\epsilon = 0.1$ and different values of δ.

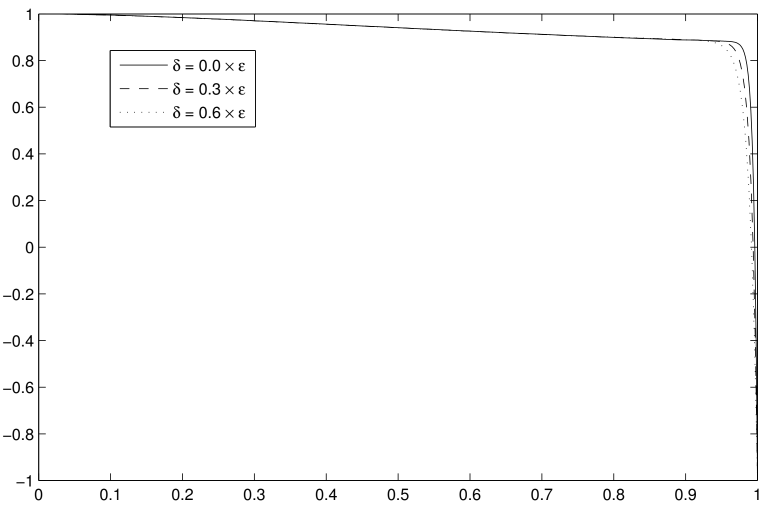

Figure 4: Approximate solutions for problem 6.2 for $\epsilon = 0.01$ and different values of δ.

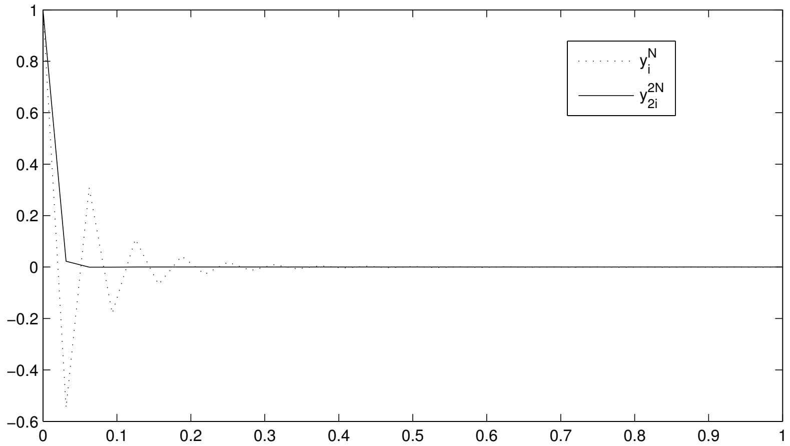

Figure 5: Approximate solutions for problem 6.1 for $\epsilon = 0.01$ and $\delta = 0.5 \times \epsilon$ with uniform mesh.

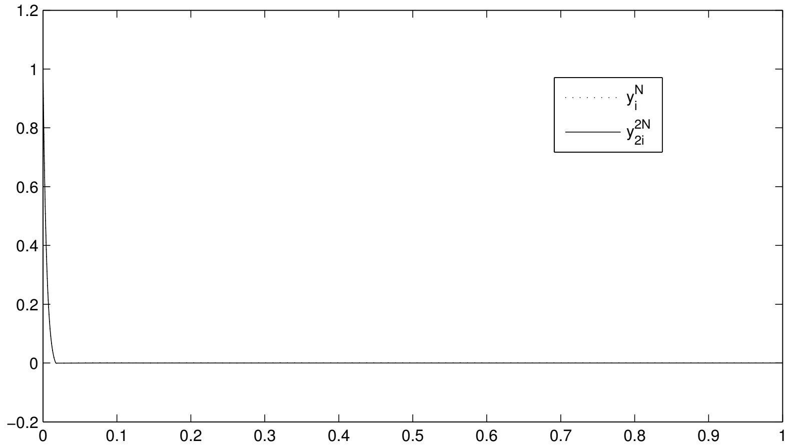

Figure 6: Approximate solutions for problem 6.1 for $\epsilon = 0.01$ and $\delta = 0.5 \times \epsilon$ with piecewise-uniform mesh.

7 Conclusion

The B-spline collocation method with Shishkin mesh, for singularly perturbed delay differential equations with negative shift in the convection as well as reaction term has been carried out. In this paper, we use Taylor's series to tackle the term containing shift. One can see from tables that the error $E^N = \max_{0<\epsilon\ll 1} E_\epsilon^N$ decreases as the mesh size decreases which shows that the fitted-mesh method is parameter-uniform. Hence, we present a parameter-uniform numerical scheme based on the fitted-mesh method for the boundary value problem (1) and (2).

The graphs of the solution of the considered examples 6.1 and 6.2, for different values of delay are plotted in Figs. 1-4, to examine the questions on the effect of delay on the boundary layer behaviour of the solution. From the results given in tables one can see that proposed method approximates the solution very well even for small values of ϵ.

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