



Rate of convergence estimates for second order elliptic eigenvalue problems on polygonal domains using spectral element methods

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Abstract: In this paper, we present rate of convergence estimates for eigenvalues and eigenvectors of elliptic differential operators on non-smooth domains using non-conforming spectral element methods. We define a class of compact operators on Banach space which is used to obtain the results. If coefficients of the differential operator are sufficiently smooth and the boundaries of the polygonal domain are piecewise analytic then exponential convergence to approximate solution is obtained.

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1 Introduction

In this paper, we present convergence estimates for eigenvalues and eigenvectors of elliptic differential operators using spectral element methods. In the era of computation the spectral methods have been proven to be faster and more accurate as compared to other alternative methods like finite element methods and finite difference methods [7, 9, 14], and the references therein. The method provides exponential convergence if the solution is sufficiently smooth, which would normally be lost when the solution shows singular behaviour at the corners of non-smooth domains.

Spectral methods for solving the problems on non-smooth domains allow only algebraic convergence [9, 14] which can be improved by use of auxiliary mapping of the form $z = \log \xi$, as proposed by Kondratiev [15]. Babuska and Guo used geometrically fine geometrical mesh in the neighborhood of each of the corner in the framework of finite element method [2, 3, 4]. To overcome the singularities at corners we also use similar kind of geometrically fine geometrical mesh and auxiliary mapping for obtaining exponential accurate solutions. Spectral methods for solving elliptic boundary value problems on non smooth domains proposed in [13, 18]. The author et al. have obtained the solution of elliptic eigenvalue problems on non smooth domains using spectral element method [6].

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In [8] Bramble and Osborn developed spectral approximation results for a class of compact operators on a Hilbert space and used the operators to obtain rate of convergence estimates for the approximation of eigenvalues and generalized eigenvectors of non-selfadjoint elliptic differential operators using Galerkin type methods. Later in [16] Osborn presented the spectral approximation results for compact operators on a Banach space. The author claimed the use of these results for various kind of approximation like, approximation of integral operators by numerical quadrature, Galerkin approximation for non-selfadjoint elliptic differential operators and formulated the results for approximation of eigenvalues and generalized eigenvectors of elliptic eigenvalue problems in terms of the norm on the underlying Banach spaces. Descloux et al. in [11, 12] developed the convergence estimates for non compact operators on Banach Spaces which were approximated by Galerkin method.

In our work, the rate of convergence of eigenvalues is reduced to the problem of rate of convergence of the approximate solution of corresponding elliptic boundary value problems as [5, 8]. Similar to [16] we define a compact operator $\mathcal{T} : V \rightarrow V$, V is a Banach space, and a family of compact operators $\mathcal{T}^P : V \rightarrow V$, such that $\mathcal{T}^P \rightarrow \mathcal{T}$ in H^1 norm, as $P \rightarrow \infty$. Here P denotes the degree of polynomials. We obtain the convergence estimates of \mathcal{T}^P , from \mathcal{T} using spectral element method. Babuska et.al [5] have shown that a compact operator \mathcal{T} satisfies the variational formulation $a(\mathcal{T}u, v) = b(u, v)$ corresponding to elliptic boundary value problems. Let λ be a nonzero eigenvalue of \mathcal{T} which satisfies variational formulation $a(u, v) = \lambda b(u, v)$, u is called an associate eigenvector, corresponding to eigenvalue problems. Similarly we concluded that (λ, u) is an eigenpair of eigenvalue problem if and only if $\lambda \mathcal{T}u = u$, i.e. (λ^{-1}, u) is an eigenpair of the compact operator \mathcal{T} [5, 10]. Further we define an integer ι which satisfy $\mathcal{N}((\lambda - \mathcal{T})^\iota) = \mathcal{N}((\lambda - \mathcal{T})^{\iota+1})$, where \mathcal{N} denotes the null space, is called the ascent of $\lambda - \mathcal{T}$. The vectors in $\mathcal{N}((\lambda - \mathcal{T})^\iota)$ are called the generalized eigenvectors of \mathcal{T} corresponding to λ [5, 8, 16].

The outline of the paper is as follows. In section 2, we define our problem on non-smooth domain and discuss the discretization of the domain. In section 3, we present our main result which shows exponential convergence for the approximation of eigenfunctions.

2 Discretization

In this section we explain the methodology and discretization of domain for elliptic boundary value problems. Consider the boundary value problem

$$\begin{aligned} \mathcal{L}u &= f \text{ in } \Omega, \\ u &= g^{[0]} \text{ on } \Gamma^{[0]}, \\ \left(\frac{\partial u}{\partial N} \right)_{\mathcal{A}} &= g^{[1]} \text{ on } \Gamma^{[1]}. \end{aligned} \quad (1)$$

Here \mathcal{L} is a strongly elliptic operator defined as

$$\mathcal{L}u = - \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left(a_{i,j}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^2 b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \quad (2)$$

where the coefficients $a_{i,j}(x) = a_{j,i}(x)$, $b_i(x)$ and $c(x)$ are analytic functions of x . Here $\left(\frac{\partial u}{\partial N} \right)_{\mathcal{A}}$ is the conormal derivative of u . Let \mathcal{A} denote the 2×2 matrix whose entries are given by

$$\mathcal{A}_{i,j}(x) = a_{i,j}(x),$$

for $i, j = 1, 2$. Then $\left(\frac{\partial u}{\partial N} \right)_{\mathcal{A}}$ is defined as

$$\left(\frac{\partial u}{\partial N} \right)_{\mathcal{A}} = \sum_{i,j=1}^2 a_{i,j} n_j \frac{\partial u}{\partial x_i}, \quad (3)$$

where $n(x) = (n_1, n_2)$ is the exterior unit normal to Γ at x . Here Γ denotes the boundary of the domain. Further, let $\Gamma = \Gamma^{[0]} \cup \Gamma^{[1]}$, $\Gamma^{[0]} = \cup_{i \in \mathcal{D}} \Gamma_i$ and $\Gamma^{[1]} = \cup_{i \in \mathcal{N}} \Gamma_i$. $\Gamma^{[0]}$ denotes the Dirichlet boundary and $\Gamma^{[1]}$ the Neumann boundary.

Let Ω be a polygon with vertices A_1, \dots, A_p and corresponding sides $\Gamma_1, \dots, \Gamma_p$, where Γ_i joins the points A_{i-1} and A_i . We divide Ω into p sub domains S^1, S^2, \dots, S^p such that each S^k contains

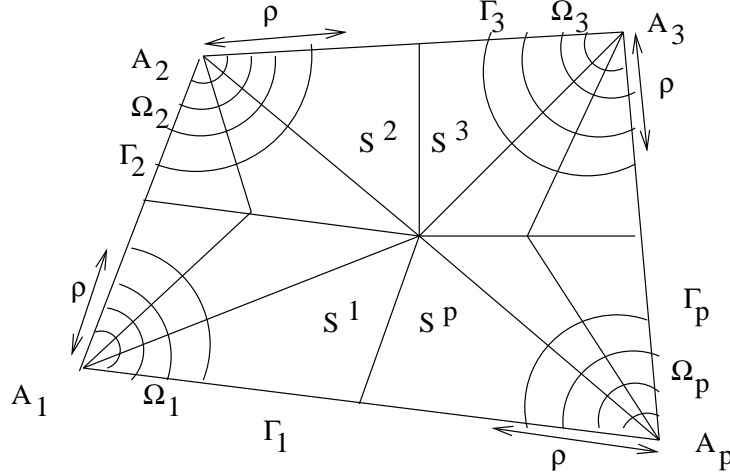


Figure 1: Discretization of the domain Ω into p non overlapping subdomains.

only the singularities at the vertex A_k , see Fig. 1. Then the sector S^k with center at A_k is defined by

$$S^k = \{ (x, y) \mid 0 < r_k < \rho, \psi_{\min}^k < \theta_k < \psi_{\max}^k \}. \quad (4)$$

Here (r_k, θ_k) are polar coordinates at vertex A_k and ρ is chosen such that $S^k \cap S^l = \emptyset$ for $k \neq l$. $\{\psi_i^k \mid i = 1, \dots, I_{k+1}\}$ is an increasing sequence of points such that $\psi_1^k = \psi_{\min}^k$ and $\psi_{I_{k+1}}^k = \psi_{\max}^k$.

Choose $\mu = (\mu_1, \dots, \mu_p)$ to be the geometric ratios with $0 < \mu_i < 1$. Let

$$\sigma_1^k = 0 \text{ and } \sigma_j^k = \rho(\mu_k)^{N+1-j} \text{ for } 2 \leq j \leq N+1,$$

and

$$\Omega_{i,j}^k = \{ (r_k, \theta_k) \mid \sigma_j^k < r_k < \sigma_{j+1}^k, \psi_i^k < \theta_k < \psi_{i+1}^k \}, \quad (5)$$

for $1 \leq i \leq I_k, 1 \leq j \leq N$.

Next, we define $\eta_j^k = \ln \sigma_j^k$ for $0 \leq j \leq N+1$. Here $\eta_0^k = -\infty$. Now we shall denote the image of $\Omega_{i,j}^k$ in (τ_k, θ_k) coordinates by $\tilde{\Omega}_{i,j}^k$, where

$$\tilde{\Omega}_{i,j}^k = \{ (\tau_k, \theta_k) \mid \eta_j^k < \tau_k < \eta_{j+1}^k, \psi_i^k < \theta_k < \psi_{i+1}^k \}, \quad (6)$$

for $1 \leq i \leq I_k, 0 \leq j \leq N$. In doing so the geometric mesh $\Omega_{i,j}^k$ for $1 \leq j \leq N$ reduces to a quasi uniform mesh. However $\tilde{\Omega}_{i,0}^k$ is a semi-infinite strip.

In the remaining part of S^k , we retain the Cartesian coordinate system (x, y) . Let

$$O^{p+1} = \{ \Omega_{i,j}^k \mid 1 \leq k \leq p, N < j \leq J_k, 1 \leq i \leq I_k \}.$$

We shall relabel the elements of O^{p+1} and write

$$O^{p+1} = \{ \Omega_l^{p+1} \mid 1 \leq l \leq L \}, \quad (7)$$

where L denotes the cardinality of O^{p+1} . Henceforth the elements in O^{p+1} are chosen to be rectangles.

We shall now define the set of non-conforming spectral element functions in a general monomial form over the sectoral neighbourhood in (τ_k, θ_k) coordinates, where $\tau_k = \ln r_k$. However away from the corners the spectral element functions are defined in (ξ, η) coordinates [6, 13]. Let $\{\{\hat{u}_{i,j}^k(\tau_k, \theta_k)\}_{i,j,k}\} \in \Pi^{N,P}$ be spectral element functions defined, by assuming that the cornermost element is constant, i.e. $\hat{u}_{i,1}^k = c_k$ for all $1 \leq i \leq I_k$ and

$$\hat{u}_{i,j}^k = \sum_{r=0}^{P_j} \sum_{s=0}^{P_j} b_{r,s} \tau_k^r \theta_k^s,$$

on $\hat{\Omega}_{i,j}^k$ for $1 \leq i \leq I_k, 2 \leq j \leq N, 1 \leq k \leq p$. Here $1 \leq P_j \leq P$.

Moreover, there is an analytic mapping M_l^{p+1} from the master square $S = (-1, 1)^2$ to Ω_l^{p+1} and hence spectral element function is define by

$$\hat{u}_l^{p+1}(\xi, \eta) = \sum_{r=0}^P \sum_{s=0}^P b_{r,s} \xi^r \eta^s.$$

Let $d(A_k, \gamma_s) = \inf_{x \in \gamma_s} \{\text{distance}(A_k, x)\}$ denote the distance between x and vertex A_k . Let $\beta = (\beta_1, \beta_2, \dots, \beta_p)$ denote a p -tuple of real numbers, $0 < \beta_i < 1, i = 1, \dots, p$.

Let $f \in H_\beta^{1,0}(\Omega)$ and we choose β_k^* so that $\beta_k^* < \beta'_k < \beta_k$ then by shift theorem [1] the solution of (1) exists in $H_{\beta'}^{3,2}(\Omega)$. Further by using the Remark 3 of [2] we can obtained the estimates

$$\|u\|_{H_{\beta'}^{3,2}(\Omega)} \leq C \|f\|_{H_\beta^{1,0}(\Omega)}. \quad (8)$$

We now define the quadratic form which is needed in the sequel.

$$\begin{aligned} & \mathcal{V}_{\text{vertices}}^{N,P} (\{\hat{u}_{i,j}^k(\tau_k, \theta_k)\}_{i,j,k}) \\ &= \sum_{k=1}^p \sum_{j=2}^N \sum_{i=1}^{I_k} (\rho \mu_k^{N+1-j})^{-2\alpha_k} \left\| \hat{\mathcal{L}}^k \hat{u}_{i,j}^k(\tau_k, \theta_k) \right\|_{0, \hat{\Omega}_{i,j}^k}^2 \\ &+ \sum_{k=1}^p \sum_{\substack{\gamma_s \subseteq \Omega^k \cup B_\rho^k, \\ \mu(\hat{\gamma}_s) < \infty}} d(A_k, \gamma_s)^{-2\alpha_k} \left(\left\| [\hat{u}^k] \right\|_{0, \hat{\gamma}_s}^2 + \left\| [\hat{u}_{\tau_k}^k] \right\|_{\frac{1}{2}, \hat{\gamma}_s}^2 + \left\| [\hat{u}_{\theta_k}^k] \right\|_{\frac{1}{2}, \hat{\gamma}_s}^2 \right) \\ &+ \sum_{l \in \mathcal{D}} \sum_{k=l-1}^l \left(|c_k|^2 + \sum_{\substack{\gamma_s \subseteq \partial \Omega^k \cap \Gamma_l, \\ \mu(\hat{\gamma}_s) < \infty}} d(A_k, \gamma_s)^{-2\alpha_k} \left(\left\| \hat{u}^k - c_k \right\|_{0, \hat{\gamma}_s}^2 + \left\| \hat{u}_{\tau_k}^k \right\|_{\frac{1}{2}, \hat{\gamma}_s}^2 \right) \right) \\ &+ \sum_{l \in \mathcal{N}} \sum_{k=l-1}^l \sum_{\substack{\gamma_s \subseteq \partial \Omega^k \cap \Gamma_l, \\ \mu(\hat{\gamma}_s) < \infty}} d(A_k, \gamma_s)^{-2\alpha_k} \left\| \left(\frac{\partial \hat{u}^k}{\partial n} \right)_{\hat{A}_k} \right\|_{\frac{1}{2}, \hat{\gamma}_s}^2. \end{aligned} \quad (9)$$

Here $\mu(\hat{\gamma}_s)$ is the length of the curve $\hat{\gamma}_s$ and $\alpha_k = 1 - \beta_k^*$.

Next, we define

$$\begin{aligned}
& \mathcal{V}_{interior}^{N,P} \left(\{\hat{u}_l^{p+1}(\xi, \eta)\}_l \right) \\
&= \sum_{l=1}^L \left\| \mathcal{L}_l^{p+1} \hat{u}_l^{p+1} \right\|_{0,S}^2 + \sum_{\gamma_s \subseteq \Omega^{p+1}} \left(\left\| [u^{p+1}] \right\|_{0,\gamma_s}^2 + \left\| [u_{x_1}^{p+1}] \right\|_{\frac{1}{2},\gamma_s}^2 + \left\| [u_{x_2}^{p+1}] \right\|_{\frac{1}{2},\gamma_s}^2 \right) \\
&+ \sum_{l \in \mathcal{D}} \sum_{\gamma_s \subseteq \partial \Omega^{p+1} \cap \Gamma_l} \left(\left\| u^{p+1} \right\|_{0,\gamma_s}^2 + \left\| \frac{\partial u^{p+1}}{\partial t} \right\|_{\frac{1}{2},\gamma_s}^2 \right) \\
&+ \sum_{l \in \mathcal{N}} \sum_{\gamma_s \subseteq \partial \Omega^{p+1} \cap \Gamma_l} \left\| \left(\frac{\partial u^{p+1}}{\partial n} \right)_{\mathcal{A}} \right\|_{\frac{1}{2},\gamma_s}^2. \tag{10}
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{V}^{N,P} \left(\{\hat{u}_{i,j}^k(\tau_k, \theta_k)\}_{i,j,k}, \{\hat{u}_l^{p+1}(\xi, \eta)\}_l \right) &= \mathcal{V}_{vertices}^{N,P} \left(\{\hat{u}_{i,j}^k(\tau_k, \theta_k)\}_{i,j,k} \right) \\
&+ \mathcal{V}_{interior}^{N,P} \left(\{\hat{u}_l^{p+1}(\xi, \eta)\}_l \right). \tag{11}
\end{aligned}$$

All the required terms in above definition are widely explained in [6].

3 Convergence estimates

In this section we shall obtain estimates for the error in approximating the eigenvalues and eigenvectors which is similar to the proof of error estimates in [6, 18].

Let \mathcal{T} be a compact operator and \mathcal{T}^P be a family of compact operators. Then we can relate the projection onto the generalized null space corresponding to eigenvalue λ to the projection associated with part of the spectrum of \mathcal{T}^P . Let λ be a nonzero eigenvalue of \mathcal{T} with algebraic multiplicity m and Γ be a circle centered at λ which lies in $\rho(\mathcal{T})$ and contains no other points of $\sigma(\mathcal{T})$. Then the spectral projection associated with (λ, \mathcal{T}) is defined by

$$E = E(\lambda) = \frac{1}{2\pi i} \int_{\Gamma} R_z(\mathcal{T}) dz. \tag{12}$$

Here E is a projection onto the space of generalized eigenvectors associated with λ and \mathcal{T} , i.e., $R(E) = \mathcal{N}((\lambda - \mathcal{T})^m)$, where $R(E)$ denotes the range of E [10].

Theorem 3.1 Let $f \in R(E)$. Then there exist constants a and k such that

$$\|(\mathcal{T} - \mathcal{T}^P) f\| \leq a e^{-kP}. \tag{13}$$

Proof: Let $f \in R(E)$ and $u = \mathcal{T}(f)$. Then \mathcal{T} is compact operator [14],[17]. Let $\hat{U}_{i,j}^k(\tau_k, \theta_k) = u(x_1, x_2) - c_k$ for $(\tau_k, \theta_k) \in \hat{\Omega}_{i,j}^k$, and $c_k = u(A_k)$. Then by using *Proposition 2.1* of [13] the estimate

$$\begin{aligned}
& \int_{\eta_j^k}^{\eta_{j+1}^k} \int_{\psi_j^k}^{\psi_{j+1}^k} \sum_{|\varepsilon| \leq m} |D_{\tau_k}^{\varepsilon_1} D_{\theta_k}^{\varepsilon_2} (\hat{U}_{i,j}^k)|^2 e^{-2\lambda_k \tau_k} d\tau_k d\theta_k \\
& \leq C(\rho \mu_k^{N+1-j})^{2\gamma_k} (Cd^{m-2}(m-2)!)^2 \tag{14}
\end{aligned}$$

holds for $1 \leq j \leq N$, where $0 < \lambda_k < \alpha_k$ and $\gamma_k < \alpha_k - \lambda_k$. Here C and d are constants independent of m .

Now consider a quadrilateral $\Omega_l^{p+1} \in \Omega^{p+1}$. Let

$$\hat{U}_l^{p+1}(\xi, \eta) = u \left(M_l^{p+1}(\xi, \eta) \right).$$

Then the following has been proved in *Lemma* 5.1 of [4].

$\hat{U}_l^{p+1}(\xi, \eta)$ is analytic on \bar{S} and hence for some constants C and d , we have

$$|D^\alpha \hat{U}_l^{p+1}(\xi, \eta)| \leq C d^m m! \quad (15)$$

for $|\alpha| = m$, $m = 1, 2, \dots$

Now using the results on approximation theory in [4, 17] there exists a polynomial $\hat{\Phi}_l^{p+1}(\xi, \eta)$ of degree P in each variable separately such that

$$\|\hat{U}_l^{p+1}(\xi, \eta) - \hat{\Phi}_l^{p+1}(\xi, \eta)\|_{2,S}^2 \leq C_s P^{-2s+8} (C d^s s!)^2 \quad (16)$$

for $1 \leq l \leq L$, where $C_s = C e^{2s}$.

Define $\hat{U}_{i,j}^k(\tau_k, \theta_k) = u(x_1, x_2) - c_k$ for $(x_1, x_2) \in \Omega_{i,j}^k$, where $c_k = u(A_k)$. Then there exists a polynomial $\hat{\Phi}_{i,j}^k(\tau_k, \theta_k)$ of degree P_j in τ_k and θ_k separately such that

$$\|\hat{U}_{i,j}^k(\tau_k, \theta_k) - \hat{\Phi}_{i,j}^k(\tau_k, \theta_k)\|_{2, \hat{\Omega}_{i,j}^k}^2 \leq C_{s_j} P_j^{-2s_j+8} (\chi_k)^{2s_j} \|\hat{U}_{i,j}^k\|_{s_j, \hat{\Omega}_{i,j}^k}^2 \quad (17)$$

where

$$\chi_k = \max \left\{ \frac{1}{2} \max_i (\Delta \psi_i^k), \frac{|\ln \mu_k|}{2}, 1 \right\}. \quad (18)$$

We now define the functional which is to be minimized.

$$\begin{aligned} & \mathcal{R}_{vertices}^{N,P} (\{\hat{u}_{i,j}^k(\tau_k, \theta_k)\}_{i,j,k}) \\ &= \sum_{k=1}^p \sum_{j=2}^N \sum_{i=1}^{I_k} (\rho \mu_k^{N+1-j})^{-2\alpha_k} \left\| \hat{\mathcal{L}}^k \hat{u}_{i,j}^k(\tau_k, \theta_k) - F^k(\tau_k, \theta_k) \right\|_{0, \hat{\Omega}_{i,j}^k}^2 \\ &+ \sum_{k=1}^p \sum_{\substack{\gamma_s \subseteq \Omega^k \cup B_\rho^k, \\ \mu(\hat{\gamma}_s) < \infty}} d(A_k, \gamma_s)^{-2\alpha_k} \left(\|\hat{u}^k\|_{0, \hat{\gamma}_s}^2 + \|\hat{u}_{\tau_k}^k\|_{\frac{1}{2}, \hat{\gamma}_s}^2 + \|\hat{u}_{\theta_k}^k\|_{\frac{1}{2}, \hat{\gamma}_s}^2 \right) \\ &+ \sum_{l \in \mathcal{D}} \sum_{k=l-1}^l \left(|c_k|^2 + \sum_{\substack{\gamma_s \subseteq \partial \Omega^k \cap \Gamma_l, \\ \mu(\hat{\gamma}_s) < \infty}} d(A_k, \gamma_s)^{-2\alpha_k} \left(\|\hat{u}^k - c_k\|_{0, \hat{\gamma}_s}^2 + \|\hat{u}_{\tau_k}^k\|_{\frac{1}{2}, \hat{\gamma}_s}^2 \right) \right) \\ &+ \sum_{l \in \mathcal{N}} \sum_{k=l-1}^l \sum_{\substack{\gamma_s \subseteq \partial \Omega^k \cap \Gamma_l, \\ \mu(\hat{\gamma}_s) < \infty}} d(A_k, \gamma_s)^{-2\alpha_k} \left\| \left(\frac{\partial \hat{u}^k}{\partial n} \right)_{\hat{A}_k} \right\|_{\frac{1}{2}, \hat{\gamma}_s}^2. \end{aligned} \quad (19)$$

Next, we define

$$\begin{aligned}
& \mathcal{R}_{interior}^{N,P} \left(\{ \hat{u}_l^{p+1}(\xi, \eta) \}_l \right) \\
&= \sum_{l=1}^L \left\| \mathcal{L}_l^{p+1} \hat{u}_l^{p+1} - F_l^{p+1}(\xi, \eta) \right\|_{0,S}^2 \\
&+ \sum_{\gamma_s \subseteq \Omega^{p+1}} \left(\left\| [u^{p+1}] \right\|_{0,\gamma_s}^2 + \left\| [u_{x_1}^{p+1}] \right\|_{\frac{1}{2},\gamma_s}^2 + \left\| [u_{x_2}^{p+1}] \right\|_{\frac{1}{2},\gamma_s}^2 \right) \\
&+ \sum_{l \in \mathcal{D}} \sum_{\gamma_s \subseteq \partial \Omega^{p+1} \cap \Gamma_l} \left(\left\| u^{p+1} \right\|_{0,\gamma_s}^2 + \left\| \frac{\partial u^{p+1}}{\partial t} \right\|_{\frac{1}{2},\gamma_s}^2 \right) \\
&+ \sum_{l \in \mathcal{N}} \sum_{\gamma_s \subseteq \partial \Omega^{p+1} \cap \Gamma_l} \left\| \left(\frac{\partial u^{p+1}}{\partial n} \right)_{\mathcal{A}} \right\|_{\frac{1}{2},\gamma_s}^2. \tag{20}
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{R}^{N,P} \left(\{ \hat{u}_{i,j}^k(\tau_k, \theta_k) \}_{i,j,k}, \{ \hat{u}_l^{p+1}(\xi, \eta) \}_l \right) &= \mathcal{R}_{vertices}^{N,P} \left(\{ \hat{u}_{i,j}^k(\tau_k, \theta_k) \}_{i,j,k} \right) \\
&+ \mathcal{R}_{interior}^{N,P} \left(\{ \hat{u}_l^{p+1}(\xi, \eta) \}_l \right).
\end{aligned}$$

Define $\hat{\Psi}_{i,0}^k(\tau_k, \theta_k) = c_k$ for $1 \leq i \leq I_k, 1 \leq k \leq p$, and $\hat{\Psi}_{i,j}^k(\tau_k, \theta_k) = \hat{\Phi}_{i,j}^k(\tau_k, \theta_k) + c_k$ for $(\tau_k, \theta_k) \in \tilde{\Omega}_{i,j}^k$ and $\hat{\Psi}_l^{p+1}(\xi, \eta) = \hat{\Phi}_l^{p+1}(\xi, \eta)$ for $(\xi, \eta) \in S$.

Now

$$\left\| \tilde{\mathcal{L}}^k \hat{\Psi}_{i,j}^k - F_{i,j}^k \right\|_{0,\tilde{\Omega}_{i,j}^k}^2 \leq \left\| \tilde{\mathcal{L}}^k (\hat{\Psi}_{i,j}^k + c_k) - \tilde{\mathcal{L}}^k (\hat{U}_{i,j}^k + c_k) \right\|_{0,\tilde{\Omega}_{i,j}^k}^2 \tag{21}$$

Hence

$$\begin{aligned}
& (\rho \mu_k^{N+1-j})^{-2\alpha_k} \left\| \tilde{\mathcal{L}}^k \hat{\Psi}_{i,j}^k - F_{i,j}^k \right\|_{0,\tilde{\Omega}_{i,j}^k}^2 \\
&\leq (\rho \mu_k^{N+1-j})^{-2\alpha_k} \left\| \tilde{\mathcal{L}}^k (\hat{\Phi}_{i,j}^k + c_k) - \tilde{\mathcal{L}}^k (\hat{U}_{i,j}^k + c_k) \right\|_{0,\tilde{\Omega}_{i,j}^k}^2 \\
&\leq C (\rho \mu_k^{N+1-j})^{-2\alpha_k} \left\| \hat{\Phi}_{i,j}^k - \hat{U}_{i,j}^k \right\|_{2,\tilde{\Omega}_{i,j}^k}^2 \\
&\leq (\rho \mu_k^{N+1-j})^{-2\alpha_k} C_{s_j} (P_j)^{-2s_j+8} (\chi_k)^{2s_j} \left\| \hat{U}_{i,j}^k \right\|_{2,\tilde{\Omega}_{i,j}^k}^2 \\
&\leq C_{s_j} (P_j)^{-2s_j+8} \left(C (\rho \mu_k^{N+1-j})^{\gamma_k} (\chi_k d)^{s_j-2} (s_j-2)! \right)^2. \quad (\text{by 14})
\end{aligned}$$

We proceed all the other terms of $\mathcal{R}_{exterior}^{N,P} \left(\{ \hat{\Psi}_{i,j}^k(\tau_k, \theta_k) \}_{i,j,k} \right)$ in same fashion and finally estimate

$$\mathcal{R}_{exterior}^{N,P} \left(\{ \hat{\Psi}_{i,j}^k(\tau_k, \theta_k) \}_{i,j,k} \right) \leq \sum_{k=1}^p \sum_{j=1}^N C_{s_j} (P_j)^{-2s_j+8} \left(C (\rho \mu_k^{N+1-j})^{\gamma_k} (\chi_k d)^{s_j} s_j! \right)^2. \tag{22}$$

Similarly it can be shown that

$$\mathcal{R}_{interior}^{N,P} \left(\{ \hat{\Psi}_l^{p+1}(\xi, \eta) \}_l \right) \leq (C_s(P))^{-2s+8} (Cd^s s!)^2. \tag{23}$$

We choose

$$\alpha j \leq P_j \leq \beta P, \quad \text{where } 0 < \alpha \text{ and } \beta \leq 1, \quad (24)$$

$$s_j \leq \Upsilon P_j, \quad \text{where } 0 < \Upsilon < 1. \quad (25)$$

Moreover $s = \Upsilon P$.

Now by using Stirling's formula

$$n! \sim \sqrt{2\pi n} e^{-n} n^n,$$

we obtain

$$\begin{aligned} & \mathcal{R}^{N,P} \left(\left\{ \hat{\Psi}_{i,j}^k(\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ \hat{\Psi}_l^{p+1}(\xi, \eta) \right\}_l \right) \\ & \leq \sum_{k=1}^p \sum_{j=1}^N C(P_j)^8 (\rho \mu_k^{N+1-j})^{2\gamma_k} (2\pi \Upsilon P_j (\chi_k \Upsilon d)^{2\Upsilon P_j}) \\ & + C(P^8 (2\pi \Upsilon P) (\Upsilon d)^{2\Upsilon P}). \end{aligned}$$

Select Υ so that $(\chi_k \Upsilon d) < 1$ for all k and $\Upsilon d < 1$. Then there exists a constant $b > 0$ such that the estimate

$$\mathcal{R}^{N,P} \left(\left\{ \hat{\Psi}_{i,j}^k(\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ \hat{\Psi}_l^{p+1}(\xi, \eta) \right\}_l \right) \leq C e^{-bP} \quad (26)$$

holds.

Let $\left(\left\{ \hat{w}_{i,j}^k(\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ \hat{w}_l^{p+1}(\xi, \eta) \right\}_l \right)$ be the space of spectral element functions, which minimizes the functional $\mathcal{R}^{N,P} \left(\left\{ \hat{\Psi}_{i,j}^k(\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ \hat{\Psi}_l^{p+1}(\xi, \eta) \right\}_l \right)$ over all i, j, k .

Hence

$$\mathcal{R}^{N,P} \left(\left\{ \hat{w}_{i,j}^k(\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ \hat{w}_l^{p+1}(\xi, \eta) \right\}_l \right) \leq C e^{-bP}. \quad (27)$$

Therefore we can conclude that

$$\mathcal{V}^{N,P} \left(\left\{ \hat{\Psi}_{i,j}^k(\tau_k, \theta_k) - \hat{w}_{i,j}^k(\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ \hat{\Psi}_l^{p+1}(\xi, \eta) - \hat{w}_l^{p+1}(\xi, \eta) \right\}_l \right) \leq C e^{-bP} \quad (28)$$

Hence by using the Stability Theorem 3.1 of [13] it can be shown that there exist constants a and k such that

$$\begin{aligned} & \sum_{k=1}^p \left(|c_k - h_k|^2 + \sum_{j=1}^N \sum_{i=1}^{I_k} (\rho \mu_k^{N+1-j})^{-2\lambda_k} \|(\hat{w}_{i,j}^k - \hat{U}_{i,j}^k)(\tau_k, \theta_k) - (h_k - a_k)\|_{2, \tilde{\Omega}_{i,j}^k}^2 \right) \\ & + \sum_{l=1}^L \|(\hat{w}_l^{p+1} - \hat{U}_l^{p+1})(\xi, \eta)\|_{2,S}^2 \leq a e^{-kP} \end{aligned} \quad (29)$$

holds.

Now we construct a set of corrections $\left(\left\{ \hat{\alpha}_{i,j}^k(\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ \hat{\alpha}_l^{p+1}(\xi, \eta) \right\}_l \right) \in \Pi^{N,P}$ such that

$$\begin{aligned} \left(\left\{ \hat{r}_{i,j}^k(\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ \hat{r}_l^{p+1}(\xi, \eta) \right\}_l \right) & = \left(\left\{ \hat{w}_{i,j}^k(\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ \hat{w}_l^{p+1}(\xi, \eta) \right\}_l \right) \\ & + \left(\left\{ \hat{\alpha}_{i,j}^k(\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ \hat{\alpha}_l^{p+1}(\xi, \eta) \right\}_l \right) \end{aligned}$$

is conforming and belongs to $H^1(\Omega)$ [6].

Define the operator $\mathcal{T}^P(f) = r$. Then \mathcal{T}^P is a compact operator since its range is finite dimensional. Then the error estimate

$$\|(\mathcal{T} - \mathcal{T}^P)(f)\|_{1,\Omega} \leq a e^{-kP}$$

holds for P large enough.

4 Conclusion

In this paper, we defined a class of compact operators and used them to obtain the exponential convergence for the approximation of eigenvalues and generalized eigenvectors of second order elliptic differential operator on non-smooth domains, where the solution exhibit the singular behaviour at corners, using non-conforming approach of spectral element methods.

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