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On the Covariance of Generalized Inverses in C^*-Algebra[1]

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Abstract: This paper gives a complete characterization of covariance set of regular elements in a C^*-algebra. Moreover, it is proved that if a and b are simply polar and regular with same range ideals, then they have the same covariance sets.

Keywords: Generalized inverse, Moore-Penrose inverse, Covariance set, C^*-algebra.

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1 Introduction

Suppose that A is a C^*-algebra with identity 1. We denote the set of invertible elements of A by A^{-1}. An element $a \in A$ is called *regular* if it has a *generalized inverse* in A, i.e. there exists $b \in A$ such that

$$a = aba.$$

In this case, ab and ba are *idempotent* elements of A.
The Moore-Penrose inverse of $a \in A$ is an element $b \in A$ such that

$$a = aba, b = bab, \quad (ab)^* = ab \text{ and } (ba)^* = ba.$$

We will denote the Moore-Penrose inverse of $a \in A$ by $a^\dagger$. If $a^\dagger$ exists, then it is unique [2, Theorem 5]. The element bab^{-1} is regular if a is regular and b is invertible [1, Theorem 2.5].
Every regular element in a C^*-algebra has a Moore-Penrose inverse (see [2, Theorem 6]).
By the uniqueness, it is clear that

$$(a^*)^\dagger = (a^\dagger)^*.$$

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Let a be an invertible element in the C^*-algebra A, its inverse a^{-1} is said to be *covariant* with respect to A^{-1}, i.e. for $b \in A^{-1}$, $(bab^{-1})^{-1} = ba^{-1}b^{-1}$.
In general, the Moore-Penrose inverse $a^\dagger$ of a is not covariant under A^{-1}. Hence the following question arises:

Question. For which $b \in A^{-1}$, the covariance condition

$$(bab^{-1})^\dagger = ba^\dagger b^{-1}$$

holds?
We shall denote this class of elements by $\mathfrak{C}(a)$, i.e.

$$\mathfrak{C}(a) = \{b \in A^{-1} : (bab^{-1})^\dagger = ba^\dagger b^{-1}\}.$$

The characterization of the covariance set $\mathfrak{C}(a)$ for the algebra of matrices was studied by H. Schwerdtfeger in [8], D. W. Robinson in [7] and A. R. Meenakshi, V. Chinnadurai in [5]. In this paper, we will characterize $\mathfrak{C}(a)$ in terms of $aa^\dagger$ and $a^\dagger a$ in a C^*-algebra.

2 Main results

Lemma 2.1. *Suppose that A is a C^*-algebra and a is a regular element in A. Then*

$$b \in \mathfrak{C}(a) \Leftrightarrow \; b^*b \in \operatorname{comm}(aa^\dagger) \cap \operatorname{comm}(a^\dagger a), \tag{1}$$

where

$$\operatorname{comm}(a) = \{x \in A : \quad xa = ax\}.$$

Proof. First suppose that $b \in \mathfrak{C}(a)$. Then $(bab^{-1})^\dagger = ba^\dagger b^{-1}$.
Since $x^\dagger x$ and $xx^\dagger$ are projections, we have

$$\left((bab^{-1})^\dagger (bab^{-1})\right)^* = (bab^{-1})^\dagger (bab^{-1}).$$

Thus $(b^{-1})^* a^\dagger a b^* = ba^\dagger ab^{-1}$. This implies that $a^\dagger ab^*b = b^*baa^\dagger$, i.e.

$$b^*b \in \text{comm } (a^\dagger a).$$

Similarly

$$b^*b \in \text{comm } (aa^\dagger).$$

For the converse, assume that $b^*b \in \operatorname{comm}(aa^\dagger) \cap \operatorname{comm}(a^\dagger a)$. Then it suffices to show that

$$(bab^{-1})^\dagger = ba^\dagger b^{-1}. \tag{2}$$

One can easily verify that the following relations hold:

$$(bab^{-1})(ba^\dagger b^{-1})(bab^{-1}) = bab^{-1}, \tag{3}$$

$$(ba^\dagger b^{-1})(bab^{-1})(ba^\dagger b^{-1}) = ba^\dagger b^{-1}. \tag{4}$$

On the other hand,

$$\begin{aligned}\left((ba^\dagger b^{-1})(bab^{-1})\right)^* &= \left((b^{-1})^* b^* b a^\dagger a b^{-1}\right)^* \\ &= ba^\dagger ab^{-1} \\ &= (ba^\dagger b^{-1})(bab^{-1}),\end{aligned} \tag{5}$$

and similarly

$$\left((bab^{-1})(ba^\dagger b^{-1})\right)^* = (bab^{-1})(ba^\dagger b^{-1}). \tag{6}$$

It follows from (3), (4), (5) and (6) that $ba^\dagger b^{-1}$ is the Moore-Penrose inverse of bab^{-1}, i.e. (2) holds. □

Lemma 2.2 *Let A be a C^*-algebra. For each regular element $a \in A$,*

$$\mathfrak{C}(a) = \mathfrak{C}(a^*) = \mathfrak{C}(a^\dagger) = \mathfrak{C}(aa^\dagger) \cap \mathfrak{C}(a^\dagger a). \tag{7}$$

Proof. Since $aa^\dagger = (aa^\dagger)^* = a^{\dagger^*} a^*$ and $a^\dagger a = (a^\dagger a)^* = a^* a^{\dagger^*}$, we have

$$b^*baa^\dagger = aa^\dagger b^*b \text{ iff } b^*ba^{\dagger^*}a^* = a^{\dagger^*}a^*b^*b,$$

and

$$b^*ba^\dagger a = a^\dagger ab^*b \ \text{ iff } \ b^*ba^*a^{\dagger^*} = a^*a^{\dagger^*}b^*b.$$

Hence

$$b \in \mathfrak{C}(a) \ \text{ iff } \ b \in \mathfrak{C}(a^*). \tag{8}$$

Moreover, $(a^\dagger)^\dagger = a$ and so

$$\begin{aligned} b^*baa^\dagger = aa^\dagger b^*b \ &\text{ iff } \ b^*b(a^\dagger)^\dagger a^\dagger = (a^\dagger)^\dagger a^\dagger b^*b, \\ b^*ba^\dagger a = a^\dagger ab^*b \ &\text{ iff } \ b^*ba^\dagger(a^\dagger)^\dagger = a^\dagger(a^\dagger)^\dagger b^*b. \end{aligned}$$

Hence

$$b \in \mathfrak{C}(a) \ \text{ iff } \ b \in \mathfrak{C}(a^\dagger). \tag{9}$$

Since $aa^\dagger$ is a self-adjoint projection, $(aa^\dagger)^\dagger = aa^\dagger$.
Also from the following relations

$$\begin{aligned} b^*baa^\dagger = aa^\dagger b^*b \ &\text{ iff } \ b^*baa^\dagger aa^\dagger = aa^\dagger aa^\dagger b^*b, \\ a^\dagger ab^*b = b^*ba^\dagger a \ &\text{ iff } \ a^\dagger aa^\dagger ab^*b = b^*ba^\dagger aa^\dagger a \end{aligned}$$

we obtain

$$\mathfrak{C}(a) = \mathfrak{C}(aa^\dagger) \cap \mathfrak{C}(a^\dagger a). \tag{10}$$

Now (7) follows from (8), (9) and (10). □

We call an element $a \in A$ *simply polar* if it has a *commuting generalized inverse*, i.e.

$$a \in a\mathrm{comm}(a)a.$$

Now by applying Lemma 2.2, we can see that if $T, S \in B(H)$ (where $B(H)$ is the C^*-algebra of all bounded linear operators on a Hilbert space H) are regular and simply polar with common *range space*, i.e. $R(T) = R(S)$, then $\mathfrak{C}(T) = \mathfrak{C}(S)$.
Associated with $a \in A$ there is the *multiplication* operator $L_a : x \mapsto ax$ from A to A. The *range ideal* of L_a is $L_a(A) = aA$.

Theorem 2.3. *Let A be a C^*-algebra. Suppose that a and b are regular and simply polar elements with the same range ideals. Then*

$$\mathfrak{C}(a) = \mathfrak{C}(b).$$

Proof. We know that $1 \in A$. So $a \in aA = bA$ and therefore $a = bc$ for some $c \in A$. Thus $a = bb^\dagger a$ which implies that $aa^\dagger = bb^\dagger aa^\dagger$. Similarly $bb^\dagger = aa^\dagger bb^\dagger$ and so $aa^\dagger = bb^\dagger$. Since a and b are simply polar, we have $a^\dagger a = b^\dagger b$ therefore

$$\begin{aligned}\mathfrak{C}(a) &= \mathfrak{C}(aa^\dagger) \cap \mathfrak{C}(a^\dagger a)\\ &= \mathfrak{C}(bb^\dagger) \cap \mathfrak{C}(b^\dagger b)\\ &= \mathfrak{C}(b). \quad \square\end{aligned}$$

Lemma 2.4. *A normal and regular element a in a C^*-algebra has the following properties*:

$$\mathfrak{C}(a) = \mathfrak{C}(a^2) = \cdots = \mathfrak{C}(a^{2n}) \text{ for all } n \in \mathbb{N}.$$

Proof. It is obvious. $\square$
The following example shows that Lemma 2.4 fails if we relax the normality condition of a.

Example 2.5. Let $a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $a^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $a^*a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $aa^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Also we have $a^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and therefore

$$\mathfrak{C}(a^2) = A^{-1} \neq \mathfrak{C}(a).$$

Collorary 2.6. *If a is a regular and normal element in a C^*- algebra A, then there exists a projection p on A such that*

$$\mathfrak{C}(a) = \mathfrak{C}(pa) = \mathfrak{C}(ap) = \mathfrak{C}(p).$$

Proof. The proof is straightforward. $\square$

Theorem 2.7. *Suppose that a is a regular element and u is a unitary element of C^*-algebra A. Then*

$$\mathfrak{C}(uau^{-1}) = u\mathfrak{C}(a)u^{-1}.$$

Proof. Let $x \in u\mathfrak{C}(a)u^{-1}$. Then $x = ubu^{-1}$ for some $b \in \mathfrak{C}(a)$. Since u is unitary, we have $(uau^{-1})^\dagger = ua^\dagger u^{-1}$. One can easily see that

$$\begin{aligned}x^*x(uau^{-1})(uau^{-1})^\dagger &= (uau^{-1})(uau^{-1})^\dagger x^*x,\\ (uau^{-1})^\dagger(uau^{-1})x^*x &= x^*x(uau^{-1})^\dagger(uau^{-1}).\end{aligned}$$

Therefore $x \in \mathfrak{C}(uau^{-1})$, thus $u\mathfrak{C}(a)u^{-1} \subseteq \mathfrak{C}(uau^{-1})$. Also we have $u^{-1}\mathfrak{C}(uau^{-1})u \subseteq \mathfrak{C}(a)$ and so $\mathfrak{C}(uau^{-1}) \subseteq u\mathfrak{C}(a)u^{-1}$. $\square$

We know that if A is a C^*-algebra with no nonzero nilpotent element, then for any regular element a, we have $\mathfrak{C}(a) = A^{-1}$. So $\mathfrak{C}(a)$ has a group structure. In general, $\mathfrak{C}(a)$ is not a group, see for instance [5].
Now a question arises: Which subset of $\mathfrak{C}(a)$ has a group structure?

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