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Finite Difference, Finite element and B-Spline Collocation Methods Applied to Two Parameter Singularly Perturbed Boundary Value Problems[1]

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Abstract: The objective of this paper is to present a comparative study of fitted-mesh finite difference method, Ritz-Galerkin finite element method and B-spline collocation method for a two-parameter singularly perturbed boundary value problems. Due to the small parameters ϵ and μ, the boundary layers arise. We have taken a piecewise-uniform fitted-mesh to resolve the boundary layers and shown that fitted-mesh finite difference method has almost first order parameter-uniform convergence, Ritz-Galerkin finite element method has almost second order parameter-uniform convergence and B-spline collocation method has second order parameter-uniform convergence. Numerical experiments support these theoretical results.

Keywords: Singular perturbation, Finite difference method, Two parameter, Boundary layers, Ritz-Galerkin method, Shishkin mesh, B-Spline collocation method

Mathematics Subject Classification: 34D15, 65L10, 65N30, 34B08

1 Introduction

Consider the following class of two parameter singularly perturbed boundary value problems

$$Ly(x) \equiv -\epsilon y''(x) - \mu a(x)y'(x) + b(x)y(x) = f(x), \quad x \in \Omega = (0,1) \tag{1}$$

with

$$y(0) = \alpha, \quad y(1) = \beta, \tag{2}$$

where two small parameters $0 < \epsilon \ll 1$ and $0 < \mu \ll 1$, are such that $\mu^2/\epsilon \to 0$ as $\epsilon \to 0$. The functions $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently smooth with $a(x) \geq a^\star > 0$, and $b(x) \geq b^\star > 0$ for $x \in [0,1]$. These kind of problem arises in transport phenomena in chemistry,

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biology, chemical reactor theory, lubrication theory and dc motor theory [1, 2, 3, 4, 5]. This problem encompasses both the reaction-diffusion problem when $\mu = 0$ and the convection-diffusion problem when $\mu = 1$. It is well-known that standard numerical methods are unsuitable for singularly perturbed problems and fail to give accurate solutions. There is a vast literature dealing with numerical methods for convection-diffusion and associated problems [6, 7]. A good number of research papers can be found in the litrature for single parameter convection-diffusion and reaction-diffusion problems but only a very few authors have discussed two-parameter singular perturbation problems [8, 9, 10, 11, 12, 13, 14].

The nature of the two-parameter problem was asymptotically examined by O'Malley [15, 16, 17]. He discussed the two cases $\mu^2/\epsilon \to 0$ as $\epsilon \to 0$ and $\epsilon/\mu^2 \to 0$ as $\mu \to 0$ and established sufficient conditions for convergence of the solution of the boundary value problem (1) and (2). O'Malley gave the asymptotic solution of two parameter problem in 1967 and after a decade Shishkin and Titov [11], used an exponentially fitted finite difference scheme on a uniform mesh for solving (1) and (2), showed that method is convergent and independent of ϵ and N. Instead of using exponential fitting, the use of layer adapted meshes have been more popular. Linβ and Roos [10], consider linear two-parameter singularly perturbed convection-diffusion problem and use the simple upwind-difference scheme on Shishkin mesh to establish almost first-order convergence, independently of the parameters ϵ and μ. Roos and Uzelac [9], also consider the two parameter singularly perturbed boundary value problem and use stream-line diffusion finite element method (SDFEM) on properly chosen Shishkin mesh and proved almost second-order pointwise convergence uniformly with respect to the parameters ϵ and μ.

The comparative study of finite difference method (FDM), finite element method (FEM) and finite volume method (FVM) for a self-adjoint two point boundary value problems has been discussed in [18]. In addition to this paper, Caglar et al. [19] compared B-spline collocation method to the FDM, FEM and FVM for the same regular self-adjoint two point boundary value problems. Kadalbajoo et al. [20], extended the results for a more general singularly perturbed boundary value problems via: fitted mesh finite difference method, finite element method and B-spline collocation method. In this paper, we have compared the fitted mesh finite difference method, finite element method and B-spline collocation method for a two parameter singularly perturbed two point boundary value problems.

The rest of the paper is organized as follows: In Section 2, we discuss a priori estimates of the continuous problem and show the boundedness of the exact solution and prove the maximum principle. In Section 3, we introduce an appropriate Shishkin mesh. In Section 4, the derivation of finite difference method has been discussed. In Section 5, the Ritz-Galerkin finite element method has been discussed and we have shown almost second order ϵ-uniform convergence. In Section 6, the derivation of the B-spline collocation method has been discussed [29]. In Section 7, numerical results and comparison of approximate solutions based on FDM, FEM and B-spline collocation methods have been presented. Finally, Section 8 contains conclusion. Throughout the paper, we use C as a generic positive constant independent of ϵ, μ and N, which may take different values at different places.

2 A Priori Estimates of the Continuous Problem

The construction of a layer-adapted mesh as well as the analysis of the given methods require information about the behaviour of derivatives of the exact solution. To describe the exponential layers at $x = 0$ and $x = 1$ we use the characteristic equation

$$-\epsilon\lambda^2(x) - \mu a(x)\lambda(x) + b(x) = 0.$$

It has two real solutions $\lambda_0(x) < 0$ and $\lambda_1(x) > 0$ which characterize the layers at $x = 0$ and $x = 1$, respectively. Let

$$\mu_0 = -\max_{x\in[0,1]} \lambda_0(x) \quad \text{and} \quad \mu_1 = \min_{x\in[0,1]} \lambda_1(x).$$

The quantity $\lambda_0(x) < 0$ describes the boundary layer at $x = 0$, while $\lambda_1(x) > 0$ characterises the layer at $x = 1$. The situations of two external layers are characterised by $\mu^2 \ll \epsilon$ or $\mu^2/\epsilon \to 0$ as $\epsilon \to 0$, which implies that $\mu_0 \approx \mu_1 \approx \sqrt{\frac{b^*}{\epsilon}}$ and we have the layers similar to the reaction-diffusion case $\mu = 0$. and when $\epsilon \ll \mu^2$ or $\epsilon/\mu^2 \to 0$ as $\mu \to 0$, which implies that $\mu_0 = \sqrt{\frac{b^*}{a^*\mu}}$ but $\mu_1 = \sqrt{\frac{a^*\mu}{\epsilon}}$ is much larger than μ_0. So the layer at $x = 1$ is much stonger than the layer at $x = 0$.

We first establish *a priori* bounds for the solution and its derivatives. The differential operator L satisfies the following maximum principle:

Lemma 2.1 *Let π be any sufficiently smooth function satisfying $\pi(0) \geq 0$ and $\pi(1) \geq 0$ and let $L\pi(x) \geq 0,\ \forall\, x \in \Omega$ implies that $\pi(x) \geq 0,\ \forall\, x \in \bar{\Omega}$.*

Proof. The proof is by contradiction. If possible suppose that there is a point $x^* \in [0,1]$ such that $\pi(x^*) < 0$ and $\pi(x^*) = \min_{x\in\bar{\Omega}} \pi(x)$. It is clear from the given conditions $x^* \notin \{0,1\}$, therefore $\pi'(x^*) = 0$ and $\pi''(x^*) \geq 0$. Thus we have

$$\begin{aligned} L\pi(x)\mid_{x=x^*} &= -\epsilon\pi''(x) - \mu a(x)\pi'(x) + b(x)\pi(x)\mid_{x=x^*} \\ &= -\epsilon\pi''(x^*) - a(x^*)\pi'(x^*) + b(x^*)\pi(x^*) < 0, \end{aligned}$$

a contradiction. It follows that $\pi(x^*) \geq 0$ and so $\pi(x) \geq 0\ \forall\, x \in \bar{\Omega}$.

Lemma 2.2 *For any $0 < p < 1$ we have upto the a certain order q that depends on the smoothness of data*

$$|y^k(x)| \leq C\left\{1 + \mu_0^k e^{-p\mu_0 x} + \mu_1^k e^{-p\mu_1(1-x)}\right\}, \quad \textit{for} \quad 0 \leq k \leq q, \tag{3}$$

Proof. We shall prove (3) by induction with respect to k. For $k = 0$ the estimate follows immediately from the maximum principle.
Since

$$(Ly)' = Ly' - \mu a'y' + b'y,$$

Then by differentiating the original equation k times, we get

$$L_k y^{(k)} = -\epsilon y^{(k+2)} - \mu a y^{(k+1)} + b_k y^{(k)} = g_k \quad \text{for} \quad k = 1, .. \tag{4}$$

where

$$b_k = b - k\mu a', \quad g_k = g'_{k-1} - b'_{k-1} y^{(k-1)} \quad \text{for} \quad k = 1, 2, 3, ...$$

The $(k+1)^{th}$ derivative satisfies the equation

$$L_{k+1} y^{k+1} = g_{k+1}$$

with

$$|g_{k+1}| \leq C \left\{ 1 + \mu_0^k e^{-p\mu_0 x} + \mu_1^k e^{-p\mu_1(1-x)} \right\}.$$

In the first step of the proof we assume that we have already proved the inequalities

$$|y^{(k+1)}(0)| \leq C\mu_0^{k+1} \quad \text{and} \quad |y^{(k+1)}(1)| \leq C\mu_1^{k+1}. \tag{5}$$

We shall bound $y^{(k+1)}$ using (4), (5) and the following barrier function

$$\Phi = C\left(1 + \mu_0^{k+1} w_{0,p}(x) + \mu_1^{k+1} w_{1,p}(x)\right),$$

with

$$w_{0,p}(x) = e^{-p\mu_0 x} \quad \text{and} \quad w_{1,p}(x) = e^{-p\mu_1(1-x)}.$$

It is sufficient to show that $L_{k+1}w_{0,p} \geq \alpha_0 w_{0,p}$ and $L_{k+1}w_{1,p} \geq \alpha_1 w_{1,p}$ with some $\alpha_0, \alpha_1 > 0$. Since

$$\begin{aligned} L_{k+1}w_{0,p} &= \left\{-\epsilon p^2 \mu_0^2 + \mu p \mu_0 a + b + O(\mu)\right\} w_{0,p} \\ &\geq \left\{p(-\epsilon \mu_0^2 + \mu a \mu_0) + b + O(\mu)\right\} w_{0,p} \\ &\geq \left\{p(-\epsilon \lambda_0^2(x) - \mu a(x)\lambda_0(x)) + b(x) + O(\mu)\right\} w_{0,p} \\ &= \left\{b(x)(1-p) + O(\mu)\right\} w_{0,p}, \end{aligned}$$

for any $0 < p < 1$, we have the desired property if μ is small enough. Similarly, we estimate $L_{k+1}w_{1,p}$.

In the next step we shall prove (5) starting from

$$L_k y^{(k)} = g_k \quad \text{with} \quad |y^{(k)}(0)| \leq C\mu_0^k, \quad |y^{(k)}(1)| \leq C\mu_1^k.$$

Setting

$$y^{(k)} = v + v_0 \quad \text{with} \quad v_o = y^{(k)}(0)(1-x),$$

we obtain for v the partially homogeneous problem

$$L_k v = g_k + \mu a v_0' - b_k v_0,$$
$$v(0) = 0, \quad v(1) = y^k(1).$$

Now we bound v by a barrier function of the type

$$\Phi = C_1(1 + e^{-p\mu_0 x} - 2e^{-\theta\mu_0 x})\mu_0^k + C_2(e^{-p\mu_1(1-x)} - e^{-p\mu_1})\mu_1^k,$$

where $\theta > 0$ is large enough. It is easy to check that $\Phi(0) = 0$ and $\Phi(1) \geq v(1)$ if C_2 is sufficiently large. Since $L_k e^{-\theta\mu_0 x} \leq 0$ and $L_k e^{-p\mu_0 x} \geq 0, L_k e^{-p\mu_1(1-x)} \geq 0$, if θ and p are chosen adequately, Φ majorizes v if C_1 is sufficiently large.

Then the definition of the derivatives gives

$$|v'(0)| \leq C\mu_0^{k+1} + C\mu_1^{k+1} e^{-p\mu_1} \leq C\mu_0^{k+1}$$

Similarly we can prove the second estimate in (5).

From (3) we obtain immediately the existence of a Shishkin decomposition (see ref. [21], p. 892)

Lemma 2.3 *The solution $y(x)$ of (1) and (2) has the representation $y(x) = S(x) + E_0(x) + E_1(x)$, where*

$$|S^k(x)| \leq C \quad \textit{for} \quad 0 \leq k \leq q,$$

and

$$\begin{aligned} |E_0^k(x)| &\leq C\mu_0^k e^{-p\mu_0 x} \quad \textit{for} \quad 0 \leq k \leq q, \\ |E_1^k(x)| &\leq C\mu_1^k e^{-p\mu_1(1-x)} \quad \textit{for} \quad 0 \leq k \leq q. \end{aligned}$$

3 Mesh Selection Strategy

Its known that on an equidistant mesh no scheme can attain convergence at all mesh points uniformly in ϵ, μ, unless its coefficients have an exponential property. Therefore, unless we use a specially chosen mesh we will not be able to get parameter-uniform convergence at all the mesh points [7]. If a priori information is available about the exact solution then we could go for a highly non-uniform grid like Bakhvalov [23], Vulanovic [24] and Gartland [25]. The simplest possible non-uniform mesh, namely a piecewise-uniform mesh proposed by Shishkin [26], is sufficient for the construction of an parameter-uniform method. It is fine near layers but coarser otherwise. We do not claim that these piecewise-uniform meshes are optimal in any sense. It is attractive because of its simplicity and adequate for handling a wide variety of singularly perturbed problems [27] and the drawback to Shishkin meshes is that one should have a priori knowledge about the location and nature of the layers.
To obtain the discrete counterpart of the singularly perturbed problem (1) and (2), first we discretize the domain $\bar{\Omega} = [0, 1]$ as as $\bar{\Omega}^N = \{0 = x_0 < x_1 < .... < x_N = 1\}$ and the piecewise-uniform mesh is defined as follows. Let N be a positive integer and multiple of 4; to define the mesh points, we divide the interval $[0, 1]$ into three subintervals $[0, \tau_1]$, $[\tau_1, 1 - \tau_2]$ and $[1 - \tau_2, 1]$, where the transition parameter is given by

$$\tau_1 = \min\left\{\frac{1}{4}, \frac{2\ln N}{\mu_0}\right\}, \quad \tau_2 = \min\left\{\frac{1}{4}, \frac{2\ln N}{\mu_1}\right\}$$

Then, we place $N/4$ mesh points each in $[0, \tau_1]$, $[1 - \tau_2, 1]$ and $N/2$ mesh points in $[\tau_1, 1 - \tau_2]$ respectively. We will denote the step size in interval $[0, \tau_1]$

$$h_1 = \frac{4\tau_1}{N}$$

$$x_i = x_{i-1} + h_1 \quad \text{for } i = 1, 2, \ldots, N/4.$$

Where $x_0 = 0$. Also, we will denote the step size in interval $[\tau_1, 1 - \tau_2]$

$$h_2 = \frac{2(1 - \tau_1 - \tau_2)}{N}$$

$$x_i = x_{i-1} + h_2 \quad \text{for } i = N/4 + 1, \ldots, 3N/4.$$

Similarly, the step size in interval $[1 - \tau_2, 1]$

$$h_3 = \frac{4\tau_2}{N}$$

$$x_i = x_{i-1} + h_3 \quad \text{for } i = 3N/4 + 1, \ldots, N.$$

4 Finite Difference Method and Convergence

Let $\bar{\Omega}^N : 0 = x_0 < x_1 < .... < x_N = 1$ be an arbitrary mesh with local step sizes $h_i = x_i - x_{i-1}$. Let $Y = Y_i$ be any given function defined on the computational mesh, we shall approximate the first-order and second-order derivatives at the grid point x_i as follows: The forward and backward divided difference operators are

$$[D^+Y]_i = \frac{Y_{i+1} - Y_i}{h_{i+1}}, \quad \text{and} \quad [D^-Y]_i = \frac{Y_i - Y_{i-1}}{h_i},$$

respectively. The central difference operator δ is given by

$$[\delta^2 Y]_i = \frac{2}{h_i + h_{i+1}} \left[D^+ Y_i - D^- Y_i\right],$$

then the fitted mesh finite difference scheme for (1) and (2) is defined by

$$[L^N Y]_i := -\epsilon[\delta^2 Y]_i - \mu a_i [D^+ Y]_i + b_i Y_i = f_i \quad \text{for} \quad i = 1 \dots N-1,$$

$$\frac{-2\epsilon}{h_i + h_{i+1}} \left[D^+ Y_i - D^- Y_i\right] - \mu a_i [D^+ Y_i] + b_i Y_i = f_i \quad \text{for} \quad i = 1 \dots N-1,$$

which leads to a trigonal system

$$E_i Y_{i-1} + F_i Y_i + G_i Y_{i+1} = f_i, \quad i = 1, 2, 3, \dots N-1, \tag{6}$$

where

$$E_i = \frac{-2\epsilon}{h_i(h_i + h_{i+1})}, \quad F_i = \frac{2\epsilon}{h_i h_{i+1}} + \frac{\mu a_i}{h_{i+1}} + b_i, \quad G_i = \frac{-2\epsilon}{h_{i+1}(h_i + h_{i+1})} - \frac{\mu a_i}{h_{i+1}},$$

$a(x_i) = a_i$, $b(x_i) = b_i$ and $f(x_i) = f_i$. The equation $L^N Y_i = f_i$ for $i = 1, 2, \dots N-1$ may be considered as a system of $N-1$ linear equations with unknowns $\{Y_i\}_1^{N-1}$. It is easy to see that the matrix is diagonally dominant and has non-positive off diagonal entries. Hence, the matrix is an irreducible M-matrix, so the scheme is stable and free from non-physical oscillations and also has positive inverse[28].

Theorem 4.1 *Let $a, b, f \in C^2[0,1]$. Let y be the solution of (1) and (2) and Y its numerical approximation by (6)on the Shishkin mesh. Then*

$$|y_i - Y_i| \leq \begin{cases} CN^{-1} & \text{for } i = N/4, \dots, 3N/4, \\ CN^{-1} \ln N & \text{otherwise.} \end{cases}$$

Proof. For the proof of the above theorem readers can see [21].

5 Finite Element Method and Convergence

The theoretical Ritz-Galerkin method is a powerful one. The approximations are very good and the method is easy to implement. To obtain better and better approximations, we choose hat functions as basis function, due to which we get a matrix of great many zeros such as band matrices.

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, & \text{for } x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h_{i+1}}, & \text{for } x \in [x_i, x_{i+1}], \\ 0, & \text{for } x \notin [x_{i-1}, x_{i+1}]. \end{cases} \tag{7}$$

The finite element subspace $\bar{V}_1(\bar{\Omega}^N)$ is the space of standard piecewise linear polynomials on a piecewise uniform fitted mesh $\bar{\Omega}^N$. The piecewise uniform fitted mesh $\bar{\Omega}^N = \{x_i\}_0^N$ condensing at the boundary points $x = 0$ and $x = 1$. The piecewise linear polynomials are continuous on $\bar{\Omega}$ and are required to vanish at the boundary points $x = 0$ and $x = 1$. It is clear that $\bar{V}_1(\bar{\Omega}^N)$ is a subspace of $H_0^1(\bar{\Omega})$. The standard basis for $\bar{V}_1(\bar{\Omega}^N)$ is $\{\phi_i\}_1^{N-1}$, where ϕ_i is the usual hat function for the mesh point x_i.

$$Ly(x) \equiv -\epsilon y''(x) - \mu a(x) y'(x) + b(x) y(x) = f(x), \quad x \in [0,1]$$

with

$$y(0) = \alpha, \quad y(1) = \beta.$$

Let

$$l(x) = \beta \frac{x-a}{b-a} + \alpha \frac{b-x}{b-a}$$

and let $y(x) = u(x) + l(x)$. Then $u(x)$ satisfies the problem.

$$-\epsilon u''(x) - \mu a(x)u'(x) + b(x)u(x) = F(x), \quad x \in [0,1] \tag{8}$$

with

$$u(0) = 0, \quad u(1) = 0.$$

Where

$$F(x)) = f(x) + \left(\epsilon \frac{\beta - \alpha}{b-a}\right)' + \mu a(x) \left(\frac{\beta-\alpha}{b-a}\right) - b(x)l(x).$$

Divide the interval $[0,1]$ up to $N+1$ subintervals, so there will be N interior grid points given by $x_i = x_{i-1} + h_i; \quad i = 1, 2, 3....,N$. Set $x_0 = 0$ and $x_{N+1} = 1$. Construct the hat functions ϕ_i, $i = 1, 2, ..., N$ and Let

$$\bar{u}(x) = \sum_{i=1}^{N} c_i \phi_i(x) \tag{9}$$

Simplify the equations (8) and (9), we get

$$\sum_{j=1}^{N} \left\{ \begin{array}{c} \int_0^1 (\epsilon \phi_i'(x)\phi_j'(x) - \mu a(x)\phi_i(x)\phi_j'(x) \\ +b(x)\phi_i(x)\phi_j(x))dx \end{array} \right\} c_j = \int_0^1 F(x)\phi_i(x)dx \tag{10}$$

$i = 1, 2, 3.....,N$. Above system of equations give a tridiagonal matrix in $c_j's$. Once the $c_j's$ have been determined from (10), the approximation $\bar{y}(x)$ to the solution, $y(x)$ of (1) and (2) is given by

$$\bar{y}(x) = \bar{u}(x) + l(x).$$

To proof a parameter-uniform estimate of $\bar{u}(x) - u(x)$ in the maximum norm where $\bar{u}(x)$ is the $\bar{V_1}(\bar{\Omega}^N)$-interpolant of the exact solution $u(x)$ of (8).

Theorem 5.1 *Let $\bar{u}(x)$ be the $\bar{V_1}(\Omega^N)$ interpolant of the solution $u(x)$ of (8) on the fitted mesh $\bar{\Omega}^N$. Then*

$$\sup_{0<\epsilon,\mu\ll 1} \|\bar{u}(x) - u(x)\|_{\bar{\Omega}} \leq CN^{-2}(\ln N)^2$$

where C is a constant independent of ϵ and μ.

Proof. The estimate is obtained on each subinterval $\Omega_i = (x_{i-1}, x_i)$ separately. Let any function g on Ω_i

$$\bar{g}(x) = g_{i-1}(x)\phi_{i-1}(x) + g_i(x)\phi_i(x)$$

and so it is obvious that, on Ω_i

$$\bar{g}(x) \leq \max_{\Omega_i} g_(x)[\phi_{i-1}(x) + \phi_i(x)]$$

Taking maximum norm both side, we get

$$|\bar{g}(x)| \le \max_{\Omega_i} \left|g_(x)\right| \tag{11}$$

and by appropriate Taylor expansions it is easy to see that

$$|\bar{g}(x) - g(x)| \le Ch_i^2 \max_{\Omega_i} |g^{''}(x)| \tag{12}$$

From (12) and Lemma 2.2, on Ω_i,

$$\begin{aligned} |\bar{u}(x) - u(x)| &\le Ch_i^2 \max_{\Omega_i} |u^{''}(x)| \\ &\le Ch_i^2 \left\{1 + \mu_0^2 e^{-p\mu_0 x} + \mu_1^2 e^{-p\mu_1(1-x)}\right\} \\ &\le Ch_i^2 \left\{1 + \mu_0^2 + \mu_1^2\right\} \end{aligned} \tag{13}$$

Also, using Lemma 2.3 and equation (12) on Ω_i,

$$\begin{aligned} |\bar{u}(x) - u(x)| &\le |\bar{S} + \bar{E_0} + \bar{E_1} - S - E_0 - E_1| \\ &\le |\bar{S} - S| + |\bar{E_0} - E_0| + |\bar{E_1} - E_1| \\ &\le Ch_i^2 \max_{\Omega_i} |S^{''}(x)| + 2\max_{\Omega_i} |E_0(x)| + 2\max_{\Omega_i} |E_1(x)| \\ &\le C(h_i^2 + e^{-\mu_0 x_i} + e^{-\mu_1(1-x_i)}) \end{aligned} \tag{14}$$

Case 1: The argument now depends on whether $\frac{2\ln N}{\mu_0} \ge 1/4$ and $\frac{2\ln N}{\mu_1} \ge 1/4$. In this case $\mu_0 \le C\ln N$ and $\mu_1 \le C\ln N$. Then the results follows at once from (13)

$$|\bar{u}(x) - u(x)| \le CN^{-2}(\ln N)^2$$

Case 2: When $\tau_1 = \frac{2\ln N}{\mu_0}$ and $\tau_2 = \frac{2\ln N}{\mu_1}$. Suppose that i satisfies $1 \le i \le N/4$ and $3N/4 \le i \le N$. Then $h_i = \frac{4\tau_1}{N} = \frac{CN^{-1}\ln N}{\mu_0}$ and $h_i = \frac{4\tau_2}{N} = \frac{CN^{-1}\ln N}{\mu_1}$ respectively. Now from (13)

$$|\bar{u}(x) - u(x)| \le CN^{-2}(\ln N)^2$$

If i satisfied $N/4 < i \le 3N/4$. Then $\tau_1 \le x_i$ and $x_i \le 1 - \tau_2$ or $\tau_2 \le 1 - x_i$ and so

$$e^{-\mu_0 x_i} \le e^{-\mu_0 \tau_1} = e^{-2\ln N} = N^{-2}, \quad \text{since } \tau_1 = \frac{2\ln N}{\mu_0}$$

$$e^{-\mu_1(1-x_i)} \le e^{-\mu_0 \tau_2} = e^{-2\ln N} = N^{-2}, \quad \text{since } \tau_2 = \frac{2\ln N}{\mu_1}$$

Using this in (14) gives the required result.

6 B-Spline Collocation Method

We define $L_2(\bar{\Omega})$ a vector space of all the square integrable function on $\bar{\Omega}$, and X be the linear subspace of $L_2(\bar{\Omega})$. For $i = -1, 0, 1, 2, ..., N+1$, we define cubic B-splines by the following relationships [29]

$$B_i(x) = \frac{1}{h^3}\begin{cases} (x - x_{i-2})^3, & x_{i-2} \le x \le x_{i-1}, \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x_{i-1} \le x \le x_i, \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x_i \le x \le x_{i+1}, \\ (x_{i+2} - x)^3, & x_{i+1} \le x \le x_{i+2}, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Let $\pi = \{x_0, x_1, \ldots, x_N\}$ be the partition of $\bar{\Omega}$. We introduce four additional knots $x_{-2} < x_{-1} < x_0$ and $x_{N+2} > x_{N+1} > x_N$, then π become $\pi = \{x_{-2} < x_{-1} < x_0 < x_1, \ldots, x_N < x_{N+1} < x_{N+2}\}$. From the above equation (15) we can simply check that each of the function $B_i(x)$ is twice continuously differentiable on the entire real line. For the values of $B_i(x_j)$, $B_i'(x_j)$ and $B_i''(x_j)$ readers can refer [29].

Let $\Omega1 = \{B_{-1}, B_0, B_1, ..., B_{N+1}\}$ and let $\Phi_3(\pi)$=span $\Omega1$. The function $\Omega1$ are linearly independent on $[0, 1]$, thus $\Phi_3(\pi)$ is $(N + 3)$-dimensional. In fact $\Phi_3(\pi) \subseteq_{subpace} X$. Let L be a linear operator whose domain is X and whose range is also in X. Now we define

$$S(x) = \sum_{i=-1}^{N+1} \gamma_i B_i(x) \tag{16}$$

where γ_i are unknown real coefficients. Here we have introduced two extra cubic B-splines, B_{-1} and B_{N+1} to satisfy the boundary conditions. Therefore we have

$$LS(x_i) = f(x_i), \qquad 0 \le i \le N, \tag{17}$$

and

$$S(x_0) = \alpha, \qquad S(x_N) = \beta. \tag{18}$$

On solving the collocation equations (17) with (18) and putting the values of B-spline functions $B_i's$ and its derivatives at mesh points, we obtain a system of $(N + 1)$ linear equations in $(N + 3)$ unknowns

$$\begin{aligned}(-6\epsilon + 3\mu a_i h + b_i h^2)\gamma_{i-1} &+ (12\epsilon + 4b_i h^2)\gamma_i \\ &+ (-6\epsilon - 3\mu a_i h + b_i h^2)\gamma_{i+1} = h^2 f_i, \quad 0 \le i \le N.\end{aligned} \tag{19}$$

The given boundary conditions become

$$\gamma_{-1} + 4\gamma_0 + \gamma_1 = \alpha, \tag{20}$$

and

$$\gamma_{N-1} + 4\gamma_N + \gamma_{N+1} = \beta. \tag{21}$$

Thus the Eqs. (19), (20) and (21) lead to a $(N + 3) \times (N + 3)$ system with $(N + 3)$ unknowns $\gamma = (\gamma_{-1}, \gamma_0, \gamma_1, ..., \gamma_{N+1})^t$. Now eliminating γ_{-1} from first equation of (19) and (20) we find

$$(36\epsilon + 12\mu a_0 h)\gamma_0 + (6\mu a_0 h)\gamma_1 = h^2 f_0 - \alpha(-6\epsilon + 3\mu a_0 h + b_0 h^2) \tag{22}$$

Similarly, eliminating γ_{N+1} from the last equation of (19) and (21), we find

$$(-6\mu a_N h)\gamma_{N-1} + (36\epsilon - 12\mu a_N h)\gamma_N = h^2 f_N - \beta(-6\epsilon - 3\mu a_N h + b_N h^2) \tag{23}$$

Coupling equations (22) and (23) with the second through (N-1)st equations of (19), we are lead to a system of $(N + 1)$ linear equations in $(N + 1)$ unknowns

$$A\gamma = d, \tag{24}$$

where $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_N)^t$ are the unknown real coefficients with right hand side

$$d = \{h^2 f_0 - \alpha(-6\epsilon + 3\mu a_0 h + b_0 h^2), h^2 f_1, \ldots, h^2 f_{N-1}, h^2 f_N - \beta(-6\epsilon - 3\mu a_N h + b_N h^2\}$$

and the co-efficient matrix A is given by

$$\left.\begin{array}{lll} a_{0,0} & = & 36\epsilon - 12\mu a_0 h, \\ a_{0,1} & = & -6\mu a_0 h, \\ a_{j,j-1} & = & -6\epsilon + 3\mu a_j h + b_j h^2, \quad j = 1(1)N-1, \\ a_{j,j} & = & 12\epsilon + 4 b_j h^2, \quad j = 1(1)N-1, \\ a_{j,j+1} & = & -6\epsilon - 3\mu a_j h + b_j h^2, \quad j = 1(1)N-1, \\ a_{N,N-1} & = & 6\mu a_N h, \\ a_{N,N} & = & 36\epsilon + 12\mu a_N h, \\ a_{i,j} & = & 0, \quad \forall \mid i-j \mid > 0. \end{array}\right\}$$

It can be easily seen that the matrix A is strictly diagonally dominant and hence nonsingular. Since A is nonsingular, we can solve the system $A\gamma = d$ for $\gamma_0, \gamma_1, ..., \gamma_N$ and substitute into the boundary condition (20) and (21) to obtain γ_{-1} and γ_{N+1}. Hence the method of collocation using a basis of cubic B-splines applied to problem (1.1) and (1.2) has a unique solution $S(x)$ given by (16).

For the derivation of uniform convergence, we use the following lemma [29] proved in [30].

Lemma 6.1 *The B-splines $B_{-1}, B_0, \ldots, B_{N+1}$, satisfy the inequality*

$$\sum_{i=-1}^{N+1} \mid B_i(x) \mid \leq 10, \qquad 0 \leq x \leq 1.$$

Theorem 6.1 *Let $S(x)$ be the collocation approximation from the space of cubic splines $\phi_3(\pi)$ to the solution $y(x)$ of the boundary value problem (1) and (2). If $f \in C^2[0,1]$, then the parameter-uniform error estimate is given by*

$$\sup_{0<\epsilon,\mu\ll 1} \max_{0\leq i\leq N} \mid y(x_i) - S(x_i) \mid \leq C\tilde{h}^2,$$

where C is a positive constant independent of ϵ, μ and $\tilde{h}$.

Proof. For the proof of the above theorem readers can see [8]. Where $\tilde{h} = max\{h_1, h_2, h_3\}$

7 Test Examples and Numerical Results

In this section, we apply fitted-mesh finite difference method, B-spline collocation method and finite element numerical method to the following problems.

Example 1. Consider the boundary value problem

$$-\epsilon y''(x) - \mu(1+x)y'(x) + y(x) = x,$$

with

$$y(0) = 1, \quad y(1) = 0.$$

Example 2. Consider the boundary value problem

$$-\epsilon y''(x) - \mu y'(x) + y(x) = x,$$

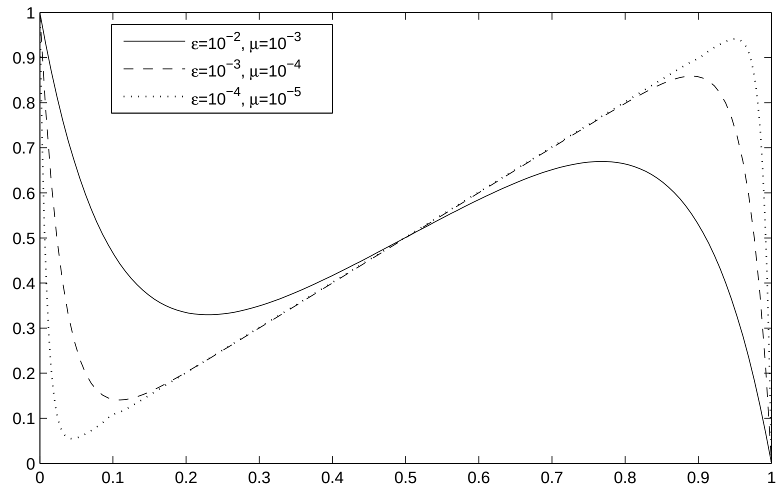

Figure 1: Solution profile of example 1. for different values of ϵ, μ.

with

$$y(0) = 1, \quad y(1) = 0.$$

Since the exact solution for considered problems are not available, the maximum absolute errors E_ϵ^N are evaluated using the double mesh principle for the given three methods

$$E_\epsilon^N = \max_{0 \le j \le N} |v_j^N - v_{2j}^{2N}|,$$

where v_j^N and v_{2j}^{2N} are the computed solutions at x_j and x_{2j} taking N and $2N$ points respectively. The numerical rates of convergence are computed using the formula

$$r_N = \log_2(E_\epsilon^N / E_\epsilon^{2N}),$$

For simple coefficients problems it is easy enough to compute analytically the coefficients in Eq. (10) while the more general problems would require numerical approximation (quadrature formula). To formulating and solving the discrete problems, FDM is less costly than the B-spline method and B-spline method is less costly than the Galerkin method even for simple coefficients problems. The maximum absolute errors and order of convergence for the considered examples are tabulated in the Tables 1,2,3 and 4. We have plotted two graphs to show the solution profiles for different values of ϵ, μ.

8 Conclusion

Three numerical methods are developed to solve a two parameter singularly perturbed boundary value problems on a piecewise-uniform mesh to resolve the boundary layers. The methods are shown to be uniformly convergent i.e., independent of mesh parameters and perturbation parameter ϵ, μ. As is evident from the numerical and theoretical results, B-spline collocation methods gives the better approximation and order of convergence than the other two methods. Ritz-Galerkin finite element method give the better approximation and order of convergence than the finite

Table 1: Comparison of maximum absolute errors and order of convergence of example 1 for $\mu = 10^{-4}$.

ϵ	N=128			N=256			N=512		
	FDM	FEM	BS	FDM	FEM	BS	FDM	FEM	BS
10^{-1}	2.3641E-03 0.9999	3.9655E-06 1.9903	4.0386E-06 2.0001	1.1821E-03 1.0000	9.9800E-07 1.9801	9.8517E-07 2.0000	5.9105E-04 0.9999	2.5295E-07 1.9583	2.4629E-07 2.0000
10^{-2}	3.8548E-03 1.0002	7.0093E-05 1.9958	6.9830E-05 2.0007	1.9270E-03 1.0000	1.7574E-05 1.9906	1.7521E-05 2.0000	9.6345E-04 1.0000	4.4221E-06 1.9811	4.3804E-06 2.0000
10^{-3}	4.0221E-03 1.0371	7.0802E-04 2.0112	6.9993E-04 2.0076	1.9599E-03 1.0047	1.7563E-04 1.9991	1.7652E-04 2.0018	9.7672E-04 1.0002	4.3932E-05 1.9858	4.4072E-05 2.0002
10^{-4}	6.2962E-03 1.0505	5.5556E-03 1.2039	1.3252E-03 1.2064	3.0398E-03 1.0523	2.4116E-03 1.2206	5.7425E-04 1.2611	1.4657E-03 1.0542	1.0348E-03 1.2112	2.3959E-04 1.3084

Table 2: Comparison of maximum absolute errors and order of convergence of example 1 for $\epsilon = 10^{-2}$.

μ	N=128			N=256			N=512		
	FDM	FEM	BS	FDM	FEM	BS	FDM	FEM	BS
10^{-3}	3.8579E-03	7.0792E-05	7.0951E-05	1.9282E-03	1.8311E-05	1.7731E-05	9.6394E-04	4.8959E-06	4.4327E-06
	1.0005	1.9508	2.0005	1.0002	1.9030	2.0000	1.0000	1.8150	2.0000
10^{-4}	3.8548E-03	7.0093E-05	7.0120E-05	1.9270E-03	1.7574E-05	1.7521E-05	9.6345E-04	4.4221E-06	4.3804E-06
	1.0002	1.9958	2.0007	1.0000	1.9906	2.0000	1.0000	1.9811	2.0000
10^{-5}	3.8545E-03	7.0034E-05	7.0037E-05	1.9269E-03	1.7505E-05	1.7500E-05	9.6342E-04	4.3793E-06	4.3752E-06
	1.0002	2.0002	2.0007	1.0000	1.9989	1.9999	1.0000	1.9981	2.0000
10^{-6}	3.8545E-03	7.0028E-05	7.0029E-05	1.9269E-03	1.7498E-05	1.7498E-05	9.6342E-04	4.3751E-06	4.3747E-06
	1.0002	2.0007	2.0007	1.0000	1.9998	1.9999	1.0000	1.9998	2.0000

Table 3: Comparison of maximum absolute errors and order of convergence of example 2 for $\mu = 10^{-4}$.

ϵ	N=128			N=256			N=512		
	FDM	FEM	BS	FDM	FEM	BS	FDM	FEM	BS
10^{-1}	2.3641E-03 1.0000	3.9408E-06 2.0000	3.9408E-06 2.0000	1.1821E-03 0.9976	9.8514E-07 2.0015	9.8514E-07 2.0015	5.9104E-04 0.9976	2.4628E-07 1.9977	2.4628E-07 1.9977
10^{-2}	3.8546E-03 0.9963	7.0125E-05 2.0021	7.0125E-05 2.0021	1.9269E-03 1.0030	1.7522E-05 1.9984	1.7522E-05 1.9984	9.6342E-04 0.9985	4.3807E-06 1.9934	4.3807E-06 1.9934
10^{-3}	4.0079E-03 1.0327	7.0989E-04 2.0041	7.0989E-04 2.0041	1.9578E-03 1.0044	1.7654E-04 2.0049	1.7654E-04 2.0049	9.7662E-04 1.0015	4.4076E-05 2.0033	4.4076E-05 2.0033
10^{-4}	6.2962E-03 1.0513	5.5428E-03 1.2009	1.3169E-03 1.2115	3.0398E-03 1.0483	2.4060E-03 1.2264	5.7046E-04 1.2600	1.4657E-03 1.0581	1.0322E-03 1.2075	2.3779E-04 1.3114

Table 4: Comparison of maximum absolute errors and order of convergence of example 2 for $\epsilon = 10^{-2}$.

μ	N=128			N=256			N=512		
	FDM	FEM	BS	FDM	FEM	BS	FDM	FEM	BS
10^{-3}	3.8558E-03	7.1006E-05	7.1006E-05	1.9273E-03	1.7745E-05	1.7745E-05	9.6355E-04	4.4362E-06	4.4362E-06
	1.0000	2.0041	2.0041	1.0015	1.9951	1.9951	1.0000	2.0000	2.0000
10^{-4}	3.8546E-03	7.0125E-05	7.0125E-05	1.9269E-03	1.7522E-05	1.7522E-05	9.6342E-04	4.3807E-06	4.3807E-06
	0.9963	2.0021	2.0021	1.0000	1.9984	1.9984	0.9985	1.9934	1.9934
10^{-5}	3.8545E-03	7.0037E-05	7.0037E-05	1.9269E-03	1.7500E-05	1.7500E-05	9.6342E-04	4.3752E-06	4.3752E-06
	0.9963	2.0000	2.0000	1.0000	1.9984	1.9984	0.9985	2.0066	2.0066
10^{-6}	3.8545E-03	7.0029E-05	7.0029E-05	1.9269E-03	1.7498E-05	1.7498E-05	9.6342E-04	4.3747E-06	4.3747E-06
	1.0000	2.0000	2.0000	1.0000	1.9984	1.9984	0.9985	2.0066	2.0066

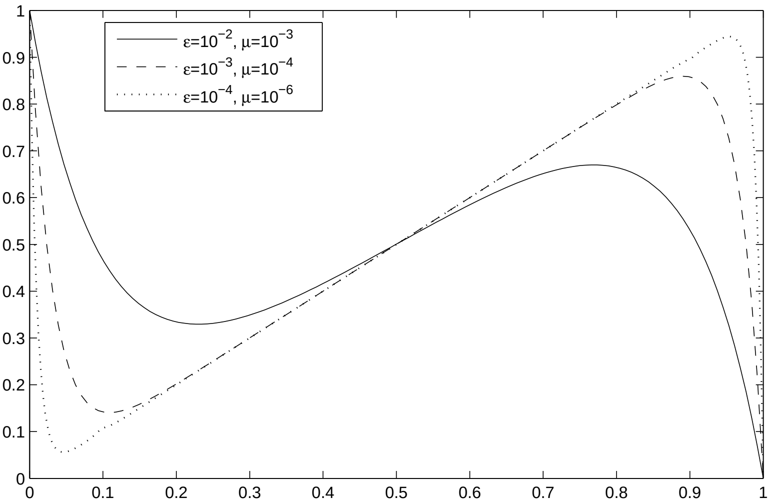

Figure 2: Solution profile of example 2. for different values of ϵ, μ.

difference method. The finite difference solution is available only at predetermined nodal points. The solution at any other point must be obtained by interpolation. On the other hand the Galerkin and collocation solutions are both given in terms of a polynomial defined over the interval $[0, 1]$. Thus, the solution is known at least in principle at every point in $[0, 1]$.

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