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A Nodal Spline Collocation Method for the Solution of Cauchy Singular Integral Equations[1]

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Dedicated to the memory of Philip Rabinowitz

Abstract: In this paper we introduce a nodal spline collocation method for the numerical solution of Cauchy singular integral equations. Uniform error bounds of the approximate solution are provided and some numerical examples are presented.

Keywords: nodal splines, singular integral equations

Mathematics Subject Classification: 65R20, 65D30, 65D07

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1 Introduction

In [14] ÷ [18] de Villiers and Rohwer introduce and analyze an arbitrary order nodal spline approximation operator with the properties of locality, interpolation at a subsequence of the spline knots and optimal order polynomial reproduction. Fundamental existence and uniqueness theorems for spline interpolation by means of additional knots, including the nodal spline case, are proved in [9].

Let $I = [a, b]$ be a given finite interval of the real line $\mathbb{R}$ and let $m \geq 3$ be the spline order. For $n \geq m - 1$, we define a partition Π_n of I by

$$\Pi_n : a = \tau_0 < \tau_1 < ... < \tau_n = b \,,$$

generally called "primary partition".

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We insert $m-2$ distinct points throughout $(\tau_\nu, \tau_{\nu+1})$, $\nu = 0, 1, ..., n-1$, obtaining the spline knots partition

$$X_n : a = x_0 < x_1 < ... < x_{(m-1)n} = b,$$

where $x_{(m-1)i} = \tau_i$, $i = 0, 1, ..., n$.

Setting

$$D_n = \max_{\substack{0\le k,j\le n-1\\ |k-j|=1}} \frac{\tau_{k+1}-\tau_k}{\tau_{j+1}-\tau_j},$$

we say that the sequence of partitions $\{\Pi_n; n = m-1, m, ...\}$ is locally uniform (l.u.) if there exists a constant $D \ge 1$ such that $D_n \le D$ for all n. We denote by h_n the norm of Π_n, i.e. $h_n = \max_{0\le i\le n-1}(\tau_{i+1} - \tau_i)$.

Now, after introducing two integers

$$i_0 = \begin{cases} \frac{1}{2}(m+1), & m \quad \text{odd} \\ \\ \frac{1}{2}m+1, & m \quad \text{even} \end{cases} \qquad \text{and} \quad i_1 = (m+1) - i_0$$

and two integer functions

$$p_\nu = \begin{cases} 0, & \nu = 0, 1, ..., i_1 - 2, \\ \nu - i_1 + 1, & \nu = i_1 - 1, ..., n - i_0, \\ n - (m-1), & \nu = n - i_0 + 1, ..., n-1, \end{cases}$$

$$q_\nu = \begin{cases} m-1, & \nu = 0, 1, ..., i_1 - 2, \\ \nu + i_0, & \nu = i_1 - 1, ..., n - i_0, \\ n, & \nu = n - i_0 + 1, ..., n-1, \end{cases}$$

we consider the set $\{\tilde{w}_i(x); i = 0, 1, ..., n\}$ of functions defined as follows:

$$\tilde{w}_i(x) = \begin{cases} l_i(x), & x \in [\tau_0, \tau_{i_1-1}], & i \le m-1, \\ s_i(x), & x \in (\tau_{i_1-1}, \tau_{n-i_0+1}), & n \ge m, \\ \bar{l}_i(x), & x \in [\tau_{n-i_0+1}, \tau_n], & i \ge n-(m-1), \end{cases}$$

where

$$l_i(x) = \prod_{\substack{k=0\\ k\ne i}}^{m-1} \frac{x-\tau_k}{\tau_i - \tau_k},$$

$$\bar{l}_i(x) = \prod_{\substack{k=0\\ k\ne n-i}}^{m-1} \frac{x-\tau_{n-k}}{\tau_i - \tau_{n-k}},$$

$$s_i(x) = \sum_{r=0}^{m-2} \sum_{j=j_0}^{j_1} \sigma_{i,r,j} B_{(m-1)(i+j)+r}(x),$$

with $j_0 = \max\{-i_0, i_1 - 2 - i\}$, $j_1 = \min\{-i_0 + m - 1, n - i_0 - i\}$. The coefficients $\sigma_{i,r,j}$ are given in [16] and the B-splines sequence is written in terms of the set $\{B_i : i = (m-1)(i_1-2), (m-1)(i_1-2)+1, ..., (m-1)(n-i_0+1)\}$ of normalized B-splines of order m defined on the set of knots X_n.

The following locality property holds:

$$s_i(x) = 0, \quad x \notin [\tau_{i-i_0}, \tau_{i+i_1}].$$

Each $\tilde{w}_i(x)$ is nodal with respect to Π_n, in the sense that

$$\tilde{w}_i(\tau_j) = \delta_{i,j}, \quad i,j = 0,1,...,n\,.$$

Therefore, being $\det[\tilde{w}_i(\tau_j)] \neq 0$, the functions $\tilde{w}_i(x), i = 0,1,...,n$, are linearly independent. Let $\mathbb{S}_{\Pi_n} = \text{span}\{\tilde{w}_i(x); i = 0,1,...,n\}$. For all $f \in \mathbb{B}(I)$, where $\mathbb{B}(I)$ is the set of real-valued functions on I, we consider the nodal spline operator $W_n : \mathbb{B}(I) \to \mathbb{S}_{\Pi_n}$, so defined

$$W_n f(x) = \sum_{i=0}^{n} f(\tau_i)\tilde{w}_i(x), \quad x \in I. \tag{1}$$

By locality property, for $0 \le \nu < n$, we can write:

$$W_n f(x) = \sum_{i=p_\nu}^{q_\nu} f(\tau_i)\tilde{w}_i(x), \quad x \in [\tau_\nu, \tau_{\nu+1}]\,.$$

In [16, 17] it is constructively proved that the following properties hold:

$$W_n f \in \mathbb{C}^{m-2}(I), \tag{2}$$

$$W_n f(\tau_i) = f(\tau_i), \quad i = 0,1,...,n, \tag{3}$$

$$W_n p = p, \quad p \in \mathbb{P}^m, \tag{4}$$

where $\mathbb{P}^m$ denotes the set of polynomials of order m (degree $\le m-1$).

The following approximation error estimate is derived in [18], for $f \in \mathbb{C}(I)$:

$$\|f - W_n f\|_\infty \le \|W_n\|\, \omega(f, mh_n), \tag{5}$$

where

$$\|W_n\| = \max_{0\le j\le n-1} \max_{\tau_j \le x \le \tau_{j+1}} \sum_{i=p_j}^{q_j} |\tilde{w}_i(x)| \le (m+1)[\sum_{\lambda=1}^{m-1} D_n^\lambda]^{m-1}$$

and $\omega(f,\delta)$ denotes the modulus of continuity on I, i.e.

$$\omega(f,\delta) = \max_{\substack{x_1,x_2\in I\\ |x_1-x_2|\le\delta}} |f(x_1) - f(x_2)|.$$

Applications to the construction of integration rules for weakly and strongly singular integrals, including Cauchy principal value (CPV) integrals in one or two dimensions, are explored also by the authors [3, 4], [6] ÷ [8], [11] ÷ [13], [19, 20, 25]. Moreover, a nodal spline collocation method is proposed and analyzed in [5], for producing the numerical solution of weakly singular Volterra integral equations of the second kind.

In this paper we present a collocation method, based on optimal nodal spline approximation, for solving the following Cauchy singular integral equation (CSIE), with constant coefficients:

$$a w_{\alpha,\beta}(x) f(x) + \frac{b}{\pi} ⨍_{-1}^{1} \frac{w_{\alpha,\beta}(t) f(t)}{t-x}\, dt + \int_{-1}^{1} w_{\alpha,\beta}(t) k(x,t) f(t) dt = g(x), \quad -1 < x < 1, \tag{6}$$

where the symbol $⨍$ means that the integral is defined in the CPV sense, $w_{\alpha,\beta}$ is the Jacobi weight function

$$w_{\alpha,\beta}(t) = (1-t)^\alpha (1+t)^\beta, \quad \alpha,\beta > -1,$$

and $k(x,t)$ is a Fredholm kernel.

We assume that f is Hölder continuous on $[-1,1]$, i.e. $f \in \mathbb{H}_\rho(C)$, where

$$\mathbb{H}_\rho(C) = \{g : g \in \mathbb{C}[-1,1], \omega(g,\delta) \leq C\delta^\rho, 0 < \rho \leq 1, \text{for some } C > 0\}.$$

The above CSIE arises in several areas as aerodynamics, elasticity, fluid and fracture mechanics. The general theory for equations of the form (6) is well developed in [23].

In particular, we consider CSIEs with index $\nu = -(\alpha+\beta) \in \{0,1\}$ ([2, 26]). If $\nu = 0$, then (6) has a unique solution; if $\nu = 1$, then an extra condition must be supplied. This condition usually takes the form

$$\int_{-1}^{1} w_{\alpha,\beta}(t) f(t) dt = c, \;\; c \in \mathbb{R}. \tag{7}$$

Our method is obtained by replacing the unknown function f by $W_n f$ in (6). If $\nu = 1$, we also replace f by the nodal spline in (7).

Then, we choose a set of collocation points $\{t_k; k = 0,1,\ldots,n-\nu\}$, different from the set X_n of knots, and obtain a system of linear equations for the unknown values $f_i^\epsilon \approx f(\tau_i)$, $i = 0,...,n$.

Once we have the approximations f_i^ϵ to $f(\tau_i)$, we can insert them into (1) and evaluate our approximation to f for any $x \in (-1,1)$.

The proposed procedure has a great flexibility: indeed, we can choose the spline space $\mathbb{S}_{\Pi_n}$, including the possibility of using the same primary knots with different values of m, and the collocation points set $\{t_k\}$.

Assuming that

$$h_n \to 0 \quad \text{as} \quad n \to \infty \tag{8}$$

and $f \in \mathbb{H}_\rho(C)$, the CPV integrals of $W_n f$, based on locally uniform partitions, converge uniformly, with respect to $x \in (-1,1)$, to the CPV integral of f [11]. This is a necessary condition to ensure the approximate solutions of (6) converge uniformly to the true solution [26] and allows to provide a uniform bound for the error of the approximate solution.

We can remark that other spline approximation operators as, for instance, quasi-interpolatory splines, reproducing polynomials up to the order of the spline, are suggested in [26] for the numerical solution of CSIEs. Nevertheless, in general, the uniform convergence of CPV integrals of such splines to the CPV integral of $f \in \mathbb{H}_\rho(C)$ cannot be proved. In order to overcome this drawback, some authors propose the evaluation of CPV integrals by other spline techniques, as either the singularity subtraction [27] or the quasi-interpolatory spline modification [10]. In [22] the Nyström method for (6), based on projector-splines, is used and the integrals are evaluated by subtracting the singularity. In [1] a Nyström type method, based on modified quasi-interpolatory splines, is introduced and the related algorithm is realized.

The method here proposed allows to directly integrate the splines, ensuring the uniform convergence for $f \in \mathbb{H}_\rho(C)$.

2 The spline collocation method

In this section we need the following notations:

$$Hf(x) = a w_{\alpha,\beta}(x) f(x) + \frac{b}{\pi} \fint_{-1}^{1} \frac{w_{\alpha,\beta}(t) f(t)}{t-x} dt, \tag{9}$$

$$Kf(x) = \int_{-1}^{1} w_{\alpha,\beta}(t)k(x,t)f(t)dt. \tag{10}$$

Rewriting equation (6), with the above notations, leads to

$$Hf + Kf = g.$$

First, we approximate f by the nodal spline $W_n f$ defined in (1) and we form the residue equation

$$r_n(x) = (HW_nf + KW_nf - g)(x) = \sum_{i=0}^{n} f(\tau_i)(H\tilde{w}_i + K\tilde{w}_i)(x) - g(x). \tag{11}$$

Then, we choose a set $\{t_k; k = 0, \ldots, n-\nu\}$ of collocation points in $(-1, 1)$, different from the set X_n of knots.

Setting $r_n(t_k) = 0$, by (11) we obtain a system of linear equations for the unknowns $f_i^\epsilon \approx f(\tau_i), i = 0, 1, \ldots, n$.

For $\nu = 1$, the system will have n equations in the $n+1$ unknowns $f_0^\epsilon, f_1^\epsilon, \ldots, f_n^\epsilon$, so that it must be augmented by the following equation, obtained approximating f by $W_n f$ in (7):

$$\sum_{i=0}^{n} f_i^\epsilon \int_{-1}^{1} w_{\alpha,\beta}(t)\tilde{w}_i(t)dt = c.$$

For $\nu = 0$, we have $n + 1$ equations in $n + 1$ unknowns.

We write the collocation system in the form:

$$A\underline{f}^\epsilon = \underline{g}\,, \tag{12}$$

where:

$$\underline{f}^\epsilon = [f_0^\epsilon, f_1^\epsilon, \ldots, f_n^\epsilon]^T,$$

$$\underline{g} = \begin{cases} [g(t_0), g(t_1), \ldots, g(t_{n-1}), c]^T, & \nu = 1, \\ [g(t_0), g(t_1), \ldots, g(t_n)]^T, & \nu = 0. \end{cases}$$

In order to define the collocation matrix entries, we introduce the following notations:

$$\overline{\mu}_i^{\alpha,\beta}(x) = \fint_{-1}^{1} \frac{w_{\alpha,\beta}(t)\tilde{w}_i(t)}{t-x}\,dt, \quad x \in (-1, 1),$$

$$\mu_i^{\alpha,\beta}(x) = K\tilde{w}_i(x)\,,$$

$$\lambda_i^{\alpha,\beta} = \int_{-1}^{1} w_{\alpha,\beta}(t)\tilde{w}_i(t)dt, \quad i = 0, 1, ..., n.$$

The elements $\{a_{k,i}\}_{k,i=0}^n$ of the matrix A in (12) can be expressed by

$$a_{k,i} = aw_{\alpha,\beta}(t_k)\tilde{w}_i(t_k) + \frac{b}{\pi}\overline{\mu}_i^{\alpha,\beta}(t_k) + \mu_i^{\alpha,\beta}(t_k), \quad k = 0, ..., n-\nu, \quad i = 0, ..., n.$$

In the case $\nu = 1$ the elements of the $(n+1)-th$ row are $a_{n,i} = \lambda_i^{\alpha,\beta}$, $i = 0, 1, ..., n$.

Therefore, we obtain the matrix entry $a_{k,i}$ by evaluating $\overline{\mu}_i^{\alpha,\beta}(t_k)$ as well as $\mu_i^{\alpha,\beta}(t_k)$ and $\lambda_i^{\alpha,\beta}$. Closed form expressions for such integrals can be derived for special choices of α, β and $k(x, t)$ [4, 25]. When closed form expressions do not exist, a numerical method must be used [21, 27].

Since the number of linear equations is equal to the number of unknown values, we can solve the system (12) for these values, provided the matrix A is nonsingular, which we assume.

Once we have the approximations f_i^ϵ to $f(\tau_i)$, we can insert them into (1), obtaining:

$$W_n f^\epsilon(x) = \sum_{i=0}^{n} f_i^\epsilon \tilde{w}_i(x)$$

and evaluate our approximation $W_n f^\epsilon$ to f for any $x \in (-1, 1)$.

3 Error estimates

The vector $\underline{f} = [f(\tau_0), f(\tau_1), ..., f(\tau_n)]^T$ of the true values satisfies

$$A\underline{f} = \underline{g} - \underline{e}, \tag{13}$$

where: $\underline{e} = [e_0, e_1, ..., e_n]^T$ is the sum of error vectors in the spline interpolation and in the numerical integration, induced by the use of nodal splines, i.e.

$$e_k = HR_n f(t_k) + KR_n f(t_k)\,, \quad k = 0, 1, ..., n - \nu,$$

with H and K defined in (9) and (10), respectively, and $R_n f = f - W_n f$. If $\nu = 1$, the last component of $\underline{e}$ is $e_n = \int_{-1}^{1} w_{\alpha,\beta}(t) R_n f(t) dt$.

Hence, from (12) and (13), we obtain the following error estimate for the spline method:

$$\|\underline{f} - \underline{f}^\epsilon\|_\infty \le \|A^{-1}\|_\infty \|\underline{e}\|_\infty. \tag{14}$$

In order to bound $\|\underline{e}\|_\infty$, we prove the following theorem.

Theorem 1 *For $f \in \mathbb{H}_\rho(C)$, let us consider a sequence $\{W_n f \in \mathbb{S}_{\Pi_n}\}$ of nodal spline approximations. Assume that the sequence $\{\Pi_n\}$ of primary partitions is l.u. and (8) holds.*

If $\rho + \gamma > 0$, where $\gamma = \min(\alpha, \beta)$, then

$$HR_n f(x) + KR_n f(x) = \begin{cases} O(h_n^\rho |\log h_n|)\,, & \gamma \ge 0 \\ O(h_n^{\rho+\gamma}), & \gamma < 0\,, \end{cases} \tag{15}$$

where, the O-term holds uniformly with respect to $x \in (-1, 1)$.

Proof. By Theorem 3.1 in [3], we know that

$$KR_n f(x) = O(h_n^\rho). \tag{16}$$

Now, we estimate $HR_n f$. By (3), we can write

$$R_n f(-1) = R_n f(1) = 0. \tag{17}$$

Taking into account the error estimate (5), since $f \in \mathbb{H}_\rho(C)$ and $\{\Pi_n\}$ is l.u., there results

$$\max_{x \in [-1,1]} |R_n f(x)| \le \bar{C} h_n^\rho\,, \quad \bar{C} > 0. \tag{18}$$

Moreover, considering that for any $u, v \in [-1, 1]$,

$$|R_n f(u) - R_n f(v)| \le |f(u) - f(v)| + |W_n f(u) - W_n f(v)|$$

and $W_n f$ is at least $\mathbb{C}^1[-1,1]$ in virtue of (2), it follows

$$R_n f \in \mathbb{H}_\rho(C_1), \quad \text{for some} \quad C_1 > 0. \tag{19}$$

Conditions (17), (18) and (19) ensure that the CPV integrals of $\{W_n f\}$ converge to the CPV integral of f uniformly in $(-1,1)$ [24] and the following estimates hold for $\rho+\gamma>0$ [12]:

$$\fint_{-1}^{1} \frac{w_{\alpha,\beta}(t) R_n f(t)}{t-x}\, dt = \begin{cases} O(h_n^\rho |\log h_n|)\,, & \gamma \geq 0 \\ O(h_n^{\rho+\gamma}), & \gamma < 0\,, \end{cases} \tag{20}$$

where, the O-term holds uniformly with respect to $x \in (-1,1)$. The result (15) follows from (5), (16) and (20). ∎

By Theorem 1 and (14), since we assume that the collocation matrix A is non singular, we can conclude

$$\|\underline{f} - \underline{f}^\epsilon\|_\infty \leq C_2 h_n^{\rho+\gamma}, \quad 0 < C_2 < \infty.$$

We remark that we consider f Hölder continuous on $[-1,1]$, because this is a general interesting case for the applications. However, if f is smoother, for instance $f \in \mathbb{C}^s[-1,1]$, $1 \leq s < m-1$ and its derivative of order s is Hölder continuous, then it is easy to show that $\|\underline{f} - \underline{f}^\epsilon\|_\infty$ is at least $O(h_n^{s-1+\rho+\gamma})$.

4 Numerical results

The above proposed spline method has been applied to solve a number of integral equations, including those here presented, for which we use uniform sets X_n of knots and several choices of collocation points different from X_n.

We denote by:

- $N = n+1$: both the primary knots number and the collocation matrix dimension;
- m: the spline order;
- E_n: the uniform norm of the error $|f(t) - W_n f^\epsilon(t)|$ computed by using a 101 uniform mesh of evaluation points in $(-1,1)$;
- $E_{\Pi_n} = max_{0 \leq i \leq n} |f(\tau_i) - W_n f^\epsilon(\tau_i)|$;
- C_N: condition number of the collocation matrix A.

Example 1

We consider the equation:

$$\frac{1}{\pi} \fint_{-1}^{1} (1-t^2)^{-\frac{1}{2}} \frac{f(t)}{t-x}\, dt = \frac{2}{\pi} \left[1 + x^2 (1-x^2)^{-\frac{1}{2}} \log \left| \frac{(1-x^2)^{\frac{1}{2}} - x + 1}{(1-x^2)^{\frac{1}{2}} + x - 1} \right| \right], \quad -1 < x < 1,$$

which has the solution $f(t) = t|t|$, if the additional condition $\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\, dt = 0$ is imposed.

Table 1 shows the rate of convergence of the method, with quadratic nodal splines ($m = 3$). The obtained results confirm the convergence properties proved in section 2. Moreover, an estimate for the collocation system condition numbers is reported for increasing values of N.

Table 1: Example 1

N	m	E_n	E_{Π_n}	C_N
6	3	7.47(-03)	2.13(-03)	3.77
12	3	1.53(-03)	4.43(-04)	7.71
36	3	1.39(-04)	4.38(-05)	23.53
82	3	2.86(-05)	8.17(-06)	53.87

Example 2

The second equation is

$$\frac{1}{\pi} \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} \frac{f(t)}{t-x}\, dt = 4x^2 - 1, \quad -1 < x < 1.$$

It has the unique solution $f(t) = 4t^3 - 3t$, with the additional condition $\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\, dt = 0$.

To this problem we apply the collocation method based on quadratic and cubic nodal splines. Table 2 provides the corresponding results.

Table 2: Example 2

N	m	E_n, E_{Π_n}	C_N
8	3	3.67(-02)	5.08
19	3	2.06(-03)	12.32
39	3	2.11(-04)	25.51
82	3	2.08(-05)	53.87
8	4	2.60(-14)	57.27

We remark that the exact solution of the equation in Example 2 belongs to $\mathbb{P}^4$ and, since (4) holds, the order of error is 10^{-14} in the case $m = 4$.

To summarize, in solving (6) by the proposed method, there are various possibilities for the choice of the spline order, of the spline knots set and of the collocation points. These features enable us to experiment with different spline spaces and different choices of sets $\{t_k\}$. Indeed, several numerical experiments can be done to evaluate the solution of (6) and get a feel for what is a good choice among the many possibilities.

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