European Society of Computational Methods in Sciences and Engineering (ESCMSE)

Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM)
vol. 3, no. 3-4, 2008, pp. 269-274
ISSN 1790–8140

Some Computational Aspects of Helly-type Theorems[1]

Phan Thanh An[2]

Institute of Mathematics,
18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam
and
The Abdus Salam International Centre for Theoretical Physics
Strada Costiera 11, 34100 Trieste, Italy

Received 21 April 2007; accepted in revised form 7 June 2008

Abstract: In this paper, we prove that, for a given positive number d, if every $n+1$ of a collection of compact convex sets in $\mathbb{E}^n$ contain a set of width d (a set of constant width d, respectively) simultaneously, then all members of this collection contain a set of constant width d_1 simultaneously, where $d_1 = d/\sqrt{n}$ if n is odd and $d_1 = d\sqrt{n+2}/(n+1)$ if n is even ($d_1 = 2d - d\sqrt{2n/(n+1)}$, respectively). This set is called common set (of constant width d_1) of the collection. These results deal with an open question raised by Buchman and Valentine in [*Croft, Falconer and Guy, Unsolved Problems in Geometry, Springer-Verlag New York, Inc. 1991, pp. 131-132*]. Moreover, given an oracle which accepts $n+1$ sets of a collection of compact convex sets in $\mathbb{E}^n$ and either returns a set of width d (a set of constant width d) contained in these sets, or reports its non-existence, we present an algorithm which determines a common set of the collection.

Keywords: Helly's theorem; set of width; set of constant width

Mathematics Subject Classification: 52A01

1 Introduction

Helly's theorem states that if $\mathcal{F}$ is a collection of convex sets in the n-dimensional Euclidean space $\mathbb{E}^n$ with the property that any $n+1$ have a common point, then all of $\mathcal{F}$ have a common point. The theorem has given rise to a vast number of variants and generalizations, known as Helly-type theorems (see [6] and [8]). Buchman and Valentine (see [4] and [5]) asked

> *What conditions are required to ensure that if every $n+1$ of a family of compact convex sets in $\mathbb{E}^n$ intersect a set of constant width d simultaneously, then all members of this family intersect a set of constant width d simultaneously?*

and

[1] Published electronically October 15, 2008
[2] E-mail:thanhan@math.ac.vn

What conditions are required to ensure that if every $n+1$ of a collection of compact convex sets in $\mathbb{E}^n$ contain a set of width d simultaneously, then all members of this collection contain a set of constant width d simultaneously?

where the width of a closed convex set is the smallest distance between parallel supporting hyperplanes of this set and a set of constant width is a compact convex set for which every two parallel support hyperplanes are at the same distance apart.

The first question was investigated in [1]. Dealing with the second one, let $\mathcal{F}$ be a collection of compact convex sets in $\mathbb{E}^2$ such that all of them contains an equilateral triangle of width d and $\mathcal{F}$ converges to the equilateral triangle. Then there does not exist any set of constant width d which is contained in the equilateral triangle. This leads to the following questions

"For a given positive number d, to find a positive number d_1 such that $d_1 < d$ and if every $n+1$ of a collection of compact convex sets in $\mathbb{E}^n$ contain a set of width d (contain a set of constant width d, respectively) simultaneously, then all members of this collection contain a set of constant width d_1 simultaneously?"

(such set of constant width d_1 is called common set of constant width d_1 of the collection). Also, if such d_1 is found, there arise questions:

"For a given oracle which accepts $n+1$ sets of a collection of compact convex sets in $\mathbb{E}^n$ and either returns a set of width d (a set of constant width d, respectively) contained in these sets, or reports its non-existence, how to determine a common set of constant width d_1?"

For a given oracle which accepts $n+1$ sets of a collection of compact convex sets in $\mathbb{E}^n$ and either returns a point in their common intersection, or reports its non-existence, [2] presented an algorithm to determine a point in the common intersection of all sets of the collection. However, until recently no such algorithm for Helly type theorems of sets of width d (sets of constant width d, respectively) had been presented.

In this paper, two Helly-type theorems which deal with these questions are presented (Propositions 2.1-2.2). Given an oracle which accepts $n+1$ sets of a collection of compact convex sets in $\mathbb{E}^n$ and either returns a set of width d (a set of constant width d, respectively) contained in these sets, or reports its non-existence, we give an algorithm which determines a common set of constant width d_1 (Algorithm 3.1).

Before starting the analysis, we recall some definitions and properties. Let $\mathcal{F}$ be a collection of compact convex sets in $\mathbb{E}^n$. A set $N \subset \mathbb{E}^n$ is a translate of a set $M \subset \mathbb{E}^n$ if $N = \{x+y : y \in M\}$ for some vector $x \in \mathbb{E}^n$.

Lemma 1.1 ([8]): *Let M be a compact convex set in $\mathbb{E}^n$. If every $n+1$ sets of $\mathcal{F}$ contain some translate of M simultaneously, then all members of $\mathcal{F}$ contain some translate of M simultaneously.*

Lemma 1.2 ([4]): *If every $n+1$ sets of $\mathcal{F}$ contain some set of width d simultaneously, then all members of $\mathcal{F}$ contain a set of width d_1 simultaneously, with $d_1 \geq d$*

A sphere $B(x,R)$ denotes the closed solid sphere in $\mathbb{E}^n$ of radius R about the centre x. Then, $B(y,R)$ is a translate of $B(x,R)$ for every $x, y \in \mathbb{E}^n$. $B(x,r)$ is called an insphere of a set M if $B(x,r) \subset M$ and r is the greatest possible. In this case r is called inradius. For a given set M of width d (a set of constant width d, respectively), there exists an insphere of M and although there may be more than one insphere, the inradius is unique (see [7] and [10]).

Lemma 1.3 ([7]): *For a given set of width d in $\mathbb{E}^n$, the inradius r of the insphere of this set satisfies*

$$r \geq \begin{cases} \frac{d}{2\sqrt{n}} & \text{if } n \text{ is odd} \\ \frac{d\sqrt{n+2}}{2(n+1)} & \text{if } n \text{ is even.} \end{cases}$$

Lemma 1.4 ([3]): *Every set of constant width d in $\mathbb{E}^n$ contains a sphere of radius*

$$r = d - d\sqrt{\frac{n}{2(n+1)}}. \tag{1}$$

Lemma 1.5 ([9]): *Suppose that $S\colon = \{x_1, x_2, \ldots, x_m\} \subset \mathbb{E}^n$. If $m \geq n+2$ then S can be partitioned into two sets S_1 and S_2 (i.e., $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$) such that* $\mathrm{conv}S_1 \cap \mathrm{conv}S_2 \neq \emptyset$.

A partition into two sets S_1 and S_2 of $S\colon = \{x_1, x_2, \ldots, x_m\}$ is called a Radon partition of S.

2 Helly-type Theorems

We now present some Helly-type theorems.

Proposition 2.1: *If every $n+1$ sets of $\mathcal{F}$ contain some set of width d simultaneously, then all members of $\mathcal{F}$ contain a sphere with radius r simultaneously satisfying*

$$r = \begin{cases} \frac{d}{2\sqrt{n}} & \text{if } n \text{ is odd} \\ \frac{d\sqrt{n+2}}{2(n+1)} & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

Proof: Consider $n+1$ arbitrary members of the collection $\mathcal{F}$. Then they contain some set of width d simultaneously. It follows from Lemma 1.3 that this set of width d contains a sphere with radius r satisfying (2). According to Lemma 1.1, all members of $\mathcal{F}$ contain some sphere with the radius r simultaneously. □

If the assumption "contain some set of width d" is replaced by "contain some set of constant width d" in Proposition 2.1, all of $\mathcal{F}$ contain a bigger sphere:

Proposition 2.2: *If every $n+1$ sets of $\mathcal{F}$ contain some set of constant width d simultaneously, then all sets of this collection contain a sphere with radius r simultaneously satisfying* (1).

Proof: It can be done exactly as the proof of Proposition 2.1 by citing Lemma 1.4 in place of Lemma 1.3 and (1) in place of (2). □

Note that the spheres in Propositions 2.1 and 2.2 are the sets of constant width $2r$. In next section, given an oracle which accepts $n+1$ sets of a collection of compact convex sets in $\mathbb{E}^n$ and either returns a set of width d (a set of constant width d) contained in these sets, or reports its non-existence, we will give an algorithm which determines a common sphere with radius r given in Proposition 2.1 (in Proposition 2.2, respectively) (Algorithm 3.1).

3 Determining a Common Set of Constant Width

First we need the following proposition:

Proposition 3.1: *Suppose that $S\colon = \{B(x_1, r), B(x_2, r), \ldots, B(x_m, r)\} \subset \mathbb{E}^n$ $(r > 0)$. If $m \geq n+2$ then S can be divided into two sets S_1 and S_2 (i.e., $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$) such that there is some sphere with radius r contained in* $\mathrm{conv}S_1 \cap \mathrm{conv}S_2$.

Proof: By Lemma 1.5, $S^*:=\{x_1, x_2, \dots, x_m\}$ can be divided into two sets S_1^* and S_2^* (i.e., $S^* = S_1^* \cup S_2^*$ and $S_1^* \cap S_2^* = \emptyset$) such that $\text{conv}S_1^* \cap \text{conv}S_2^* \neq \emptyset$. Assume without loss of generality that $S_1^* = \{x_1, x_2, \dots, x_l\}$ and $S_2^* = \{x_{l+1}, x_{l+2}, \dots, x_m\}$. Set $S_1 = \{B(x_1, r), B(x_2, r), \dots, B(x_l, r)\}$ and $S_2 = \{B(x_{l+1}, r), B(x_{l+2}, r), \dots, B(x_m, r)\}$. Take $x \in \text{conv}S_1^* \cap \text{conv}S_2^*$. It follows that

$$\begin{aligned} B(x,r) &\subset \text{conv}S_1^* \bigcap \text{conv}S_2^* + B(0,r) \\ &\subset (\text{conv}S_1^* + B(0,r)) \bigcap (\text{conv}S_2^* + B(0,r)) \\ &= \text{conv}(S_1^* + B(0,r)) \bigcap \text{conv}(S_2^* + B(0,r)) \\ &= \text{conv}S_1 \bigcap \text{conv}S_2. \end{aligned}$$

□

We also say that a partition into two sets S_1 and S_2 of $S:=\{B(x_1, r), B(x_2, r), \dots, B(x_m, r)\}$ is a Radon partition of S.

Given a collection $\mathcal{F} = \{V_1, V_2, \dots, V_m\}$ of compact convex sets in $\mathbb{E}^n$, let us assume that we have available an "oracle", say $\mathcal{O}$, which accepts as input $n+1$ sets of $\mathcal{F}$, and which gives as its output a common set of width d (a set of constant width d, respectively), or reports its non-existence, as the case may be.

For a given set P of width d (a set of constant width d, respectively), assume that c is the center of an insphere of P (the proof of Theorem 2.7.7 [10] indicates how such center c can be found). Suppose that r satisfies (2) ((1), respectively). Then, by Proposition 2.1 (Proposition 2.2, respectively), $B(c, r) \subset P$.

Algorithm 3.1:

Given a collection $\mathcal{F} = \{V_1, V_2, \dots, V_m\}$ of compact convex sets in $\mathbb{E}^n$, let us assume that we have available the oracle $\mathcal{O}$. We now determine a sphere with radius r given in Proposition 2.1 (in Proposition 2.2, respectively).

We now use the idea of the algorithm presented in [2] as follows. For each $k = 1, 2, \dots, m$, we compute a common sphere with the radius r for each collection of the form

$$\{V_1, V_2, \dots, V_k, V_{t_1}, V_{t_2}, \dots, V_{t_n}\}, \quad \{t_1, t_2, \dots, t_n\} \subset \{k+1, \dots, m\}. \tag{3}$$

Note that when $k = m - n$, this is the required common set of width d (a set of constant width d, respectively). If at any time a call to the oracle reveals that some subcollection has no common set of width d (no common set of constant width d, respectively), then the algorithm terminates .

For $k = 1$, a common set of width d (a common set of constant width d, respectively) of each of the $\binom{m-1}{n}$ families $\{V_1, V_{t_1}, V_{t_2}, \dots, V_{t_n}\}$ may be obtained directly from the oracle. In general, having found all of the common sets of width d (common sets of constant width d, respectively) for the families in (3) up to $k - 1 \geq 1$, we compute the values for k by taking a Radon partition following $n+2$ common sets of width d ($n+2$ common sets of constant width d, respectively).

- P_0, a common set of width d (a common set of constant width d, respectively) of the family $\{V_1, V_2, \dots, V_{k-1}, V_{t_1}, V_{t_2}, \dots, V_{t_n}\}$. Take the center c_0 of an insphere of P_0. Then $B(c_0, r) \subset P_0$.

- P_i, a common set of width d (a common set of constant width d, respectively) of the family $\{V_1, V_2, \ldots, V_k, V_{t_1}, V_{t_2}, \ldots, V_{t_n}\} \setminus V_{t_i}$, for $1 \leq i \leq n$. Take the center c_i of an insphere of P_i. Then $B(c_i, r) \subset P_i$.
- P_{n+1}, a common set of width d (a common set of constant width d, respectively) of the family $\{V_k, V_{t_1}, V_{t_2}, \ldots, V_{t_n}\}$. Take the center c_{n+1} of an insphere of P_{n+1}. Then $B(c_{n+1}, r) \subset P_{n+1}$.

Note that $P_0, \ldots, P_n$ have already been computed and can be looked up. The common set of width d (the common set of constant width d, respectively) P_{n+1} is obtained by a call to the oracle. Set $K_0 = V_k, K_i = V_{t_i},\ i = 1, \ldots n$, and $K_{n+1} = \bigcap_{1 \leq i \leq k-1} V_i$. It is easy to verify that $B(c_i, r) \subset \bigcap_{j \neq i} K_j$ for $0 \leq i \leq n+1$. By Proposition 3.1, the set $S = \{B(c_0, r), B(c_1, r), \ldots, B(c_{n+1}, r)\}$ can be divided into two sets S_1 and S_2, say $S_1 = \{B(c_0, r), B(c_1, r), \ldots, B(c_l, r)\}$ and $S_2 = \{B(c_{l+1}, r), B(c_{l+2}, r), \ldots, B(c_{n+1}, r)\}$, such that there is some sphere with radius r contained in conv$S_1 \cap$ convS_2. Since K_i $(0 \leq i \leq n+1)$ are convex and $B(c_i, r) \subset K_j$ with $j \neq i$, we conclude that conv$S_2 \subset K_i$ if $0 \leq i \leq l$ and conv$S_1 \subset K_i$ if $l+1 \leq i \leq n+1$. It follows that conv$S_1 \cap$ conv$S_2 \subset \bigcap_{i=0}^{n+1} K_i$. Then the sphere $B(x, r)$ is a common sphere to all the sets $V_1, V_2, \ldots, V_k, V_{t_1}, V_{t_2}, \ldots V_{t_n}$.

Note that the total number of oracle calls required by Algorithm 3.1 is $\binom{m}{n+1}$. But this number may not accurately describe the complexity of the algorithm, because it does not account for the complexity of the appeals to Proposition 3.1 or of finding the centers c_i of the inspheres of P_i.

4 Concluding Remarks

By Propositions 2.1 and 2.2, the open questions mentioned in Section 1 become

"For a given positive number d, to find a positive number d_1 such that

$$d > d_1 > \begin{cases} \frac{d}{\sqrt{n}} & \text{if } n \text{ is odd} \\ \frac{d\sqrt{n+2}}{n+1} & \text{if } n \text{ is even} \end{cases}$$

($d > d_1 > 2d - d\sqrt{\frac{2n}{n+1}}$, respectively) and if every $n+1$ of a collection of compact convex sets in $\mathbb{E}^n$ contain a set of width d (contain a set of constant width d, respectively) simultaneously, then all members of this collection contain a set of constant width d_1 simultaneously?".

In [1], some Helly-type theorems for roughly convexlike sets were presented. Similar computational aspects for such theorems should be a subject of another paper.

Acknowledgment

The author would like to express the gratitude to the referee for their valuable remarks and proposals which led to an improvement of the paper. He wishes to thank the Mathematics section of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, where part of the paper was written, for the invitation and hospitality.

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