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Optimal Stability of Bivariate Tensor Product B-bases[1]

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Abstract: We analyze some properties of bivariate tensor product bases. In particular, we obtain conditions so that a general bivariate tensor product basis is optimally stable for evaluation. Finally, we apply our results to prove the optimal stability of tensor product normalized B-bases.

Keywords: tensor product representations, normalized B-bases, optimal stability

Mathematics Subject Classification: 65D07, 65D17, 41A10, 41A15, 41A63, 41A45, 42A10

1 Introduction

Given a basis $b = (b_0, \ldots, b_n)$ of a vector space U of functions defined on ω and a function $f \in U$, there exists a unique sequence of real coefficients $(c_0, \ldots, c_n)$ such that

$$f(x) = \sum_{i=0}^{n} c_i b_i(x)$$

for all $x \in \Omega$. One practical aspect to consider in the evaluation of the function f is the stability with respect to perturbations of the coefficients, which depends on the chosen basis of the space. We want to know how sensitive a value $f(x)$ is to random perturbations of a given maximum relative magnitude ε in the coefficients $(c_0, \ldots, c_n)$ corresponding to the basis. Following [6], we can bound the corresponding perturbation $\delta f(x)$ of the change of $f(x)$ by means of a condition number

$$C_b(f, x) := \sum_{i=0}^{n} |c_i b_i(x)|\,,$$

for the evaluation of $f(x)$ in the basis b:

$$\delta f(x) \leq C_b(f, x)\varepsilon.$$

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We shall consider that a basis b is optimally stable among the nonnegative bases of its space U if there does not exist any other basis u of nonnegative functions such that $C_u(f,x) \leq C_b(f,x)$ for each function $f \in U$ evaluated at every x. In general, existence or uniqueness of optimally stable bases cannot be guaranteed. In [6] it was proved that the Bernstein basis is optimally stable among all nonnegative bases of the space of polynomials of degree less than or equal to n. In [10] it was proved that, among all nonnegative bases of its space, the B-spline basis is optimally stable for evaluating spline functions.

A basis of nonnegative functions that form a partition of the unity is called *blending*. The *collocation matrix* of a system of functions $(u_0(t),\dots,u_n(t))$ $t \in I$, at $t_0 < \cdots < t_m$ in I is given by

$$M\begin{pmatrix} u_0,\dots,u_n \\ t_0,\dots,t_m \end{pmatrix} := (u_j(t_i))_{i=0,\dots,m;j=0,\dots,n}. \tag{1}$$

A matrix is *totally positive* if all its minors are nonnegative and a system of functions is totally positive when all its collocation matrices (1) are totally positive. If the basis functions form a partition of the unity, we say that the basis is normalized. It is well-known that shape preserving representations of curves are associated with normalized totally positive bases (cf. [2], [7]).

In contrast to the space of algebraic polynomials, which possesses normalized totally positive bases on any compact interval, there are other spaces such as the space of the trigonometric polynomials (see [11]) or the spaces $\langle 1,\dots,t^{n-2},\cos(wt),\sin(wt)\rangle$, $n>1$ mentioned above, with no normalized totally positive bases on any compact interval. Finding domain intervals where we can guarantee the existence of shape preserving representations is one of the main tasks to carry out when dealing with these spaces. Another crucial task consists of finding properties of the spaces resembling the good properties of the Bernstein basis. When the space has a normalized totally positive basis, the Bernstein-like bases are provided by the normalized B-bases, which present optimal shape preserving properties (see [2], [3], [12]). A totally positive basis $(b_0,\dots,b_n)$ of a space U is a *B-basis* if for any other totally positive basis $(u_0,\dots,u_n)$ of U the matrix K of change of basis

$$(u_0,\dots,u_n) = (b_0,\dots,b_n)K$$

is totally positive (all its minors are nonnegative). In Theorem 4.2 of [3] it was proved that a space with a normalized totally positive basis always has a unique normalized B-basis. For instance, the Bernstein basis is the normalized B-basis in the case of the space of polynomials of degree at most n on a compact interval (see [2]) or the B-spline basis is the normalized B-basis in the case of the space of spline polynomials.

The following sufficient condition for B-bases corresponds to the implication (ii) implies (i) of Theorem 3.2 of [12] and is going to be used in the upcoming proofs.

Lemma 1 *Let U be a vector space of functions defined on a closed set $I \subset \mathbf{R}$ with a normalizable (respectively, normalized) totally positive basis and let $(v_0,\dots,v_n)$ be a basis (respectively, normalized basis) of U formed by nonnegative functions. Assume that for each $i = 0,1,\dots,n$, $I_i := \{t \in I_i | v_i(t) \neq 0\}$ is an interval and let $a_i \in \mathbf{R}\cup\{-\infty\}$ and $b_i \in \mathbf{R}\cup\{+\infty\}$ be the infimum and supremum, respectively, of I_i. If*

$$a_j \leq a_k, \quad b_j \leq b_k, \quad \lim_{t\to a_j^+} \frac{v_k(t)}{v_j(t)} = 0 \quad \text{and} \quad \lim_{t\to b_k^+} \frac{v_j(t)}{v_k(t)} = 0$$

whenever $j<k$, then $(v_0,\dots,v_n)$ is a B-basis (respectively, the normalized B-basis) of U.

Other properties satisfied by these bases, such as optimal stability, were shown (see [13]). Here we analyze some properties of bivariate tensor product B-bases. In particular we prove their optimal stability, which could be expected taking into account the optimal stability of the tensor product Bernstein and B-spline bases (see [8]) and that of rational Bernstein basis (see [5]).

2 Optimal stability of bivariate tensor product bases

Let $(u_0^1, \dots, u_m^1)$ and $(u_0^2, \dots, u_n^2)$ $m > 1$, $n > 1$, be bases for two spaces U_m^1 and U_n^2 of functions defined on intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. The system b of tensor-product functions defined by

$$b := (u_i^1 \otimes u_j^2)_{i=0,\dots,m,\, j=0,\dots,n} \tag{2}$$

is a basis of the space $U_m^1 \otimes V_n^2$ of bivariate functions defined on $\Omega = [a_1, b_1] \times [a_2, b_2]$. We order the elements of $b = (b_{ij})_{i=0,\dots,m, j=0,\dots,n}$ in lexicographical order.

In order to study the optimal stability of bases (2) we need to introduce some basic notations. Let U be a finite dimensional vector space of functions defined on $\Omega \subset \mathbf{R}^s$ and let $b = (b_0, \dots, b_n)$ be a basis for U. Given a function $f = \sum_{i=0}^n c_i b_i \in U$ we are interested in measures for the sensitivity of $f(x)$ to perturbations in the coefficients $c = (c_j)_{j=0,\dots,n}$ of f. If $g = \sum_{i=0}^n (1 + \delta_i) c_i b_i$ is related to f by a relative perturbation $\delta = (\delta_i)_{i=0,\dots,n}$ in c, then for any $x \in \Omega$

$$|f(x) - g(x)| = \left| \sum_{i=0}^n \delta_i c_i b_i(x) \right| \leq \|\delta\|_\infty \sum_{i=0}^n |c_i b_i(x)| \,.$$

The number

$$C_b(f, x) := \sum_{i=0}^n |c_i b_i(x)| \,, \tag{3}$$

acts as a condition number for the evaluation of f at the point x using the basis b (see [6], [10]). We can take the size of f into account and define related numbers by

$$\text{cond}(b; f, x) := \frac{C_b(f, x)}{\|f\|_\infty} = \frac{\sum_{i=0}^n |c_i b_i(x)|}{\| \sum_{i=0}^n c_i b_i \|_\infty}, \tag{4}$$

or by $\text{cond}(b; f) := \sup_{x \in \Omega} \text{cond}(b; f, x)$ and $\text{cond}(b) := \sup_{f \in U} \text{cond}(b; f)$ (see [8]).

We observe that for $f, g \in U$ as above and $x \in \Omega$

$$|f(x) - g(x)| \leq \varepsilon C_b(f, x),$$

and

$$\frac{|f(x) - g(x)|}{\|f\|_\infty} \leq \varepsilon \text{cond}(b; f, x) \leq \varepsilon \text{cond}(b; f) \leq \varepsilon \text{cond}(b),$$

where $\varepsilon = \|\delta\|_\infty$. Thus the condition numbers can be used to measure the sensitivity of f to perturbations in the coefficients.

Definition 1 *Given a set B of bases of a vector space U of functions defined on Ω, we say that a basis $b \in B$ is optimally stable for the evaluation of functions if there does not exist (up to permutation or scaling) a basis $u \in B$ such that*

$$cond(u; f, x) \leq cond(b; f, x)$$

for each function $f \in U$ evaluated at every $x \in \Omega$.

Let u, v be two bases of U. It obviously follows from the previous definitions that $\text{cond}(u; f, x) \leq \text{cond}(v; f, x)$ if and only if $C_u(f, x) \leq C_v(f, x)$. Thus, we could have defined optimal stability in terms of the condition number (3) instead of the condition number (4). Also one can easily check that if $\text{cond}(u; f, x) \leq \text{cond}(v; f, x)$ $\forall f \in U$, $\forall x \in \Omega$ then $\text{cond}(u) \leq \text{cond}(v)$.

The following auxiliary results can be found in [8]. Besides, Lemma 2 is closely related to Theorem 2.1 of [6].

Lemma 2 *Let U be a finite dimensional vector space of functions defined on $\Omega \subset \mathbf{R}^s$. Let u, v be two bases of nonnegative functions of U. Then*

$$cond(u; f, x) \leq cond(v; f, x), \quad \forall f \in U, \ \forall x \in \Omega \tag{5}$$

if and only if the matrix A such that $v = uA$ is nonnegative.

Lemma 3 *Let M be a nonsingular and nonnegative matrix such that the first and last nonzero entry of each column of M^{-1} is positive. Then $M = PD$, where D is a diagonal matrix with positive diagonal elements and P is a permutation matrix.*

Lemma 4 *Let u and b be nonnegative bases for a finite dimensional vector space U of functions defined on a set $\Omega \in \mathbf{R}^s$. Suppose that the change of basis matrix A defined by $b = uA$ is nonnegative and that the first and last nonzero element in each column of A^{-1} is positive. Then $u = b$ up to a permutation and positive scaling.*

The following result gives conditions so that a tensor-product basis is optimally stable.

Theorem 1 *Let $(u_0^1, \ldots, u_m^1)$ and $(u_0^2, \ldots, u_n^2)$ be nonnegative bases for two spaces U_m^1 and U_n^2 defined on intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively, satisfying*

$$\lim_{t \to a_k^+} \frac{u_i^k(t)}{u_j^k(t)} = 0 \ \text{ if } \ i > j \quad \text{and} \quad \lim_{t \to b_k^-} \frac{u_i^k(t)}{u_j^k(t)} = 0 \ \text{ if } \ i < j \tag{6}$$

for $k = 1, 2$. Let $b = (b_{ij})_{i=0,\ldots,m, j=0,\ldots,n}$ be the tensor-product basis

$$b_{ij}(x, y) = u_i^1(x) u_j^2(y)$$

of the space $U_m^1 \otimes U_n^2$ of bivariate functions defined on $\Omega = [a_1, b_1] \times [a_2, b_2]$. If $\tilde{b} = (\tilde{b}_{ij})_{i=0,\ldots,m, j=0,\ldots,n}$ is a basis of nonnegative functions of $U_m^1 \otimes U_n^2$ with

$$cond(\tilde{b}; f, (x, y)) \leq cond(b; f, (x, y)) \tag{7}$$

for each function $f \in U_m^1 \otimes U_n^2$ evaluated at every $(x, y) \in \Omega$, then $\tilde{b} = b$ up to permutation and positive scaling.

Proof. By Lemma 2 there exists a nonnegative matrix A such that $b = \tilde{b}A$ and, by Lemma 4 it is enough to show that the first and last nonzero element in each column of A^{-1} is positive. Fix a column β of A^{-1} and let $(c_{ij})_{i=0,\ldots,m, j=0,\ldots,n}$ be the elements of A^{-1} in this column so that

$$\tilde{b}_\beta(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} b_{ij}(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} u_i^1(x) u_j^2(y). \tag{8}$$

Let (i_1, j_1) be the index of the first nonzero c_{ij} in (8). Let us prove that $c_{i_1 j_1}$ is positive. Clearly, the function $\tilde{b}_\beta$ of (8) can be written as

$$\tilde{b}_\beta(x, y) = u_{i_1}^1(x) \left(\sum_{j=j_1}^{n} c_{i_1 j} u_j^2(y) \right) + \sum_{i=i_1+1}^{m} u_i^1(x) \left(\sum_{j=0}^{n} c_{ij} u_j^2(y) \right). \tag{9}$$

Using the nonnegativity of $\tilde{b}_\beta$ and $u^1_{i_1}$ joint with (9) and (6) we have that

$$\begin{aligned} 0 \le \lim_{x \to a_1^+} \frac{\tilde{b}_\beta(x,y)}{u^1_{i_1}(x)} &= \sum_{j=j_1}^n c_{i_1 j} u^2_j(y) + \lim_{x \to a_1^+} \sum_{i=i_1+1}^m \frac{u^1_i(x)}{u^1_{i_1}(x)} \left(\sum_{j=0}^n c_{ij} u^2_j(y) \right) \\ &= \sum_{j=j_1}^n c_{i_1 j} u^2_j(y). \end{aligned} \tag{10}$$

By (6) and the nonnegativity of $\bar{v}(y) := \sum_{j=j_1}^n c_{i_1 j} u^2_j(y)$ and $u^2_{j_1}$ we deduce that

$$0 \le \lim_{y \to a_2^+} \frac{\bar{v}(y)}{u^2_{j_1}(y)} = c_{i_1 j_1} + \sum_{j=j_1+1}^n c_{i_1 j} \lim_{y \to a_2^+} \frac{u^2_j(y)}{u^2_{j_1}(y)} = c_{i_1 j_1}.$$

Let $c_{i_2 j_2}$ be the last nonzero coefficient in (8). Applying a reasoning analogous to the proof of the positivity of $c_{i_1 j_1}$, but taking limits as $x \to b_1^-$ and $y \to b_2^-$, we may deduce that $c_{i_2 j_2} > 0$, that is, the last nonzero entry of each column of A^{-1} is positive. ■

Conditions (6) of Theorem 1 also allow us to apply the result to B-bases. As a direct consequence of the previous result and Lemma 1 we can deduce the following Corollary that guarantees the optimal stability of general bivariate tensor product B-bases. Examples of B-bases can be seen in [3] and [12]. In the next section, we focus on spaces mixing algebraic and trigonometric of hyperbolic polynomials.

Corollary 1 *Let $(u^1_0, \dots, u^1_m)$ and $(u^2_0, \dots, u^2_n)$ $m > 1$, $n > 1$, be B-bases for two spaces U^1_m and U^2_n of functions defined on intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. The tensor-product basis $b = (b_{ij})_{i=0,\dots,m, j=0,\dots,n}$ with*

$$b_{ij}(x,y) = u_{i,m}(x) v_{j,n}(y)$$

is optimally stable for the evaluation on $\Omega = [a_1, b_1] \times [a_2, b_2]$ of functions of $U_m \otimes V_n$.

3 Examples

Rational Bézier and B-spline surfaces are handy and versatile tools to represent a large variety of surfaces. However there are surfaces of interest in geometry and engineering, like the helicoid or the catenoid which can only be approximately represented by rational functions. The construction and analysis of new function spaces, more flexible than polynomials but with the same nice structural properties, constitutes an interesting new research trend in CAD (see [1], [4], [9], [15], [14]).

In [9] a unified approach to deal with these type of spaces was introduced. In addition to the well known case of algebraic polynomials, this approach includes as particular examples the spaces

$$\begin{aligned} H_n &:= \langle 1, \dots, t^{n-2}, \cosh(\omega t), \sinh(\omega t) \rangle, \\ T_n &:= \langle 1, \dots, t^{n-2}, \cos(\omega t), \sin(\omega t) \rangle, \end{aligned}$$

for $\omega \ne 0$ and $n > 1$, and piecewise functions composed by different spaces.

In order to deal with the complex geometry of real objects, we can consider the normalized B-bases of bivariate tensor product spaces obtained by considering the spaces studied in [9].

Let us consider the homogeneous linear differential equation

$$u''(t) + a_0 u(t) = 0, \quad a_0 \in \mathbf{R}. \tag{11}$$

The characteristic polynomial of (11), $p(x) = x^2 + a_0$, is an even function, and so the space U_1 of solutions of (11) is invariant under translations and reflections. Clearly,

i) If $p(x) = x^2$ then $U_1 = \text{span}\langle 1, t\rangle$.

ii) If $p(x) = x^2 - \omega^2$, $\omega \neq 0$, then $U_1 = \text{span}\langle \cosh(\omega t), \sinh(\omega t)\rangle$.

iii) If $p(x) = x^2 + \omega^2$, $\omega \neq 0$, then $U_1 = \text{span}\langle \cos(\omega t), \sin(\omega t)\rangle$.

Let us now consider the initial value problem

$$\begin{array}{l} u''(t) + a_0 u(t) = 0, \quad a_0 \in \mathbf{R}, \\ u(0) = 0,\ u'(0) = 1. \end{array} \tag{12}$$

Let S be the unique solution of (12) and z_S its first positive zero. Let us define

$$u_{0,1}(t) := S(a-t)/S(a), \quad u_{1,1}(t) := S(t)/S(a), \quad t \in [0, a],\ 0 < a < z_S. \tag{13}$$

It can be easily checked that

i) If $p(x) = x^2$, then $S(t) = t$ and $z_S = +\infty$.

ii) If $p(x) = x^2 - \omega^2$, $\omega \neq 0$, then $S(t) = \sinh(\omega t)/\omega$ and $z_S = +\infty$.

iii) If $p(x) = x^2 + \omega^2$, $\omega \neq 0$, then $S(t) = \sin(\omega t)/\omega$ and $z_S = \pi/\omega$.

Observe that, except for the case i), $1 \notin U_1$. This implies that U_1 has no normalized totally positive basis and therefore it does not possess shape preserving representations of curves. In order to obtain spaces with shape preserving representations including U_1, let us study the space of the integrals of the functions of U_1.

Let us recall that, for a given space of functions U, the space of the derivatives U' is defined by

$$U' := \{u' \mid u \in U\}.$$

For $n > 1$, the $(n+1)$-dimensional spaces U_n such that $U'_k = U_{k-1}$ for all $k = 2, \dots, n$ are:

i) If $p(x) = x^2$ then $U_n = \langle 1, \dots, t^n\rangle$.

ii) If $p(x) = x^2 - \omega^2$, $\omega \neq 0$, then $U_n = \langle 1, \dots, t^{n-2}, \cosh(\omega t), \sinh(\omega t)\rangle$.

iii) If $p(x) = x^2 + \omega^2$, $\omega \neq 0$, then $U_n = \langle 1, \dots, t^{n-2}, \cos(\omega t), \sin(\omega t)\rangle$.

In [9] we proved that for $n > 1$ and $a < z_S$ the normalized B-basis of U_n can be defined recursively by:

$$\begin{aligned} u_{0,n}(t) &:= 1 - \int_0^t \delta_{0,n-1} u_{0,n-1}(s) ds, \\ u_{i,n}(t) &:= \int_0^t \left(\delta_{i-1,n-1} u_{i-1,n-1}(s) - \delta_{i,n-1} u_{i,n-1}(s)\right) ds, \quad i = 1, \dots, n-1, \\ u_{n,n}(t) &:= \int_0^t \delta_{n-1,n-1} u_{n-1,n-1}(s) ds, \end{aligned} \tag{14}$$

for $t \in [0, a]$, where $\delta_{i,n-1} := 1/\int_0^a u_{i,n-1}(s)\, ds$, $i = 0, \dots, n-1$ and $(u_{0,1}, u_{1,1})$ is the system defined in (13).

Let us now illustrate the optimal stability (in the sense explained above) of tensor-product normalized B-bases (14) with examples.

Let us consider tensor-products of the normalized B-basis of $H_3 := \langle 1, t, \cosh t, \sinh t\rangle$ and $T_2 := \langle 1, \cos t, \sin t\rangle$ obtained by (14).

Using (14) and (13) with $S(t) = \sinh t$ we have obtained the normalized B-bases of H_3:

$$v_{3,3}(t) = \frac{t - \sinh t}{M - \sinh M}, \quad v_{0,3}(t) = v_{3,3}(M - t),$$
$$v_{2,3}(t) = \frac{(1 + \cosh M)t + \sinh(M - t) - \sinh t - \sinh M}{(1 + \cosh M)M - 2\sinh M} - v_{3,3}(t),$$
$$v_{1,3}(t) = v_{2,3}(M - t), \tag{15}$$

$t \in [0, M]$, $M < z_S = +\infty$.

Using (14) and (13) with $S(t) = \sin t$ we have obtained the normalized B-bases of T_2:

$$u_{2,2}(t) = \frac{1 - \cos t}{1 - \cos a}, \quad u_{0,2}(t) = u_{2,2}(a - t), \quad u_{1,2}(t) = 1 - u_{0,2}(t) - u_{2,2}(t), \tag{16}$$

$t \in [0, a]$, $a < z_S = \pi$.

The catenoid (catenary of revolution) and the plane are the only surfaces of revolution which are also minimal surfaces. A catenoid patch can be given by the parametric equations

$$c(u, v) = (\cosh v \cos u, \cosh v \sin u, v), \quad 0 \le u \le a, \quad 0 \le v \le M. \tag{17}$$

In order to represent a catenoid patch as a tensor-product we have computed the coefficients of the functions $\cosh t$ and t with respect to the normalized B-basis of H_3 and the coefficients of the functions $\sin t$, $\cos t$ with respect to the normalized B-basis of T_2 (see Table 1 and Table 2).

function	c_0	c_1	c_2	c_3
t	0	$\frac{M - \sinh M}{1 - \cosh M}$	$\frac{\sinh M - M\cosh M}{1 - \cosh M}$	M
$\cosh t$	1	1	$\frac{M(1 + \cosh M) - \sinh M}{\sinh M}$	$\cosh M$

Table 1. Coefficients of t and $\cosh t$ with respect to the normalized B-basis of $H_3 = \langle 1, t, \cosh t, \sinh t\rangle$ on $[0, M]$, $M > 0$.

function	c_0	c_1	c_2
$\sin t$	0	$\frac{\sin a}{1 + \cos a}$	$\sin a$
$\cos t$	1	1	$\cos ha$

Table 2. Coefficients of $\sin t$ and $\cos t$ with respect the normalized B-basis of $T_2 = \langle 1, \sin t, \cos t\rangle$ on $[0, a]$, $a < \pi$.

The catenoid patch (17) can be written as

$$c(r, s) = \sum_{i=0}^{2}\sum_{j=0}^{3} \mathbf{p}_{i,j} u_{i,2}(r) v_{j,3}(r) \tag{18}$$

$r \in [0, a]$, $s \in [0, M]$, where the control points $\mathbf{p}_{i,j}$, can be immediately obtained from the coefficients in Table 1 and Table 2 (see Figure 3).

In order to illustrate the optimal stability of the tensor product of the normalized B-bases (16) and (15) we have also considered the tensor product basis of $T_2 \bigotimes H_3$ obtained by taking the basis of T_2:

$$(\tilde{v}_{0,2}(r), \tilde{v}_{1,2}(r), \tilde{v}_{2,2}(r)) = (1, \cos r, \sin r)$$

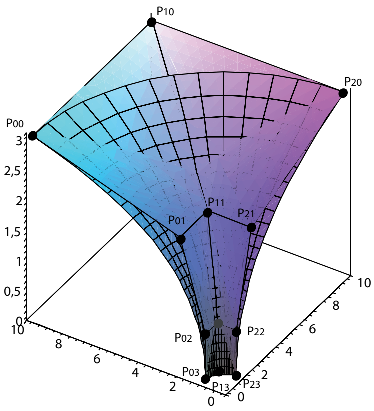

Figure 1: Catenoid patch and its control net with respect to the normalized B-basis of $T_2 \bigotimes H_3$.

and the basis of H_3:

$$(\tilde{w}_{0,3}(s), \tilde{w}_{1,3}(s), \tilde{w}_{2,3}(s), \tilde{w}_{3,3}(s)) = (1, s, \cosh s, \sinh s).$$

We have disturbed the control nets of the catenoid (17) with respect to both tensor-product bases with random disturbances $\delta_{i,j} = m_{i,j} \times 10^{-e}$ $i = 0, 1, 2$ and $j = 0, \ldots, 3$ satisfying $|m_{i,j}| \leq 1$ for $e = 3, 6$ and we have compared the catenoid (17) with the obtained surfaces.

In Figure 2 we can see the plot of the surfaces

$$e_1(r, s) = \sum_{i=0}^{2} \sum_{j=0}^{3} \delta_{i,j} u_{i,2}(r) v_{j,3}(r), \quad e_2(r, s) = \sum_{i=0}^{2} \sum_{j=0}^{3} \delta_{i,j} \tilde{v}_{i,2}(r) \tilde{w}_{j,3}(s),$$

respectively, with random disturbances satisfying $\delta_{i,j} = m_{i,j} \times 10^{-3}$ $i = 0, 1, 2$ and $j = 0, \ldots, 3$.

In Figure 3 we can see the plot of $e_1(r, s)$ and $e_2(r, s)$, respectively, with random disturbances satisfying $\delta_{i,j} = m_{i,j} \times 10^{-6}$ $i = 0, 1, 2$ and $j = 0, \ldots, 3$.

In these plots, we can immediately check that the errors obtained when representing the catenoid with the tensor-product normalized B-bases of T_2 and H_3 have a magnitude similar to the perturbations in the control net and they are considerably smaller than those obtained when representing the catenoid with the other tensor product basis.

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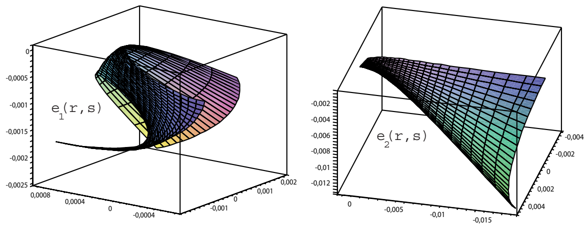

Figure 2: Plot of $e_1(r,s)$ and $e_2(r,s)$ with random disturbances satisfying $\delta_{i,j} = m_{i,j} \times 10^{-3}$ $i = 0,1,2$ and $j = 0,\ldots,3$.

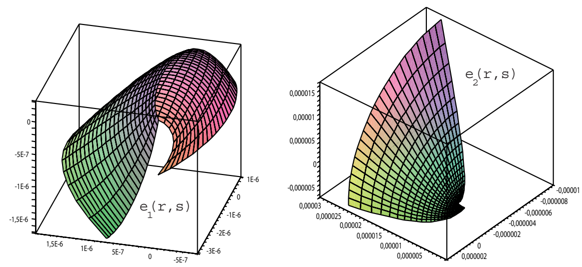

Figure 3: Plot of $e_1(r,s)$ and $e_2(r,s)$ with random disturbances satisfying $\delta_{i,j} = m_{i,j} \times 10^{-6}$ $i = 0,1,2$ and $j = 0,\ldots,3$.

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