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A Complete Orthogonal System of Spheroidal Monogenics[1]

J. Morais[2]

Freiberg University of Mining and Technology,
Institute of Applied Analysis,
Prüferstr. 9,
09596 Freiberg, Germany

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Abstract: During the past few years considerable attention has been given to the role played by monogenic functions in approximation theory. The main goal of the present paper is to construct a complete orthogonal system of monogenic polynomials as solutions of the Riesz system over prolate spheroids in $\mathbb{R}^3$. This will be done in the spaces of square integrable functions over $\mathbb{R}$. As a first step towards is that the orthogonality of the polynomials in question does not depend on the shape of the spheroids, but only on the location of the foci of the ellipse generating the spheroid. Some important properties of the system are briefly discussed.

Keywords: Quaternionic analysis, Riesz system, Ferrer's associated Legendre functions, Chebyshev polynomials, Hyperbolic functions, monogenic functions

Mathematics Subject Classification: 30G35, 31B05

1 Introduction

There are several reasons why quaternionic analysis has recently played an active part in the treatment of boundary value problems. This discipline has been recently extended to the case of initial-boundary value problems of mathematical physics given in three or four dimensions [3, 4, 6, 7, 11, 12]. It is thought to generalize the theory of holomorphic functions of one complex variable and also provides the foundations to refine the theory of harmonic functions in higher dimensions. The rich structure of this function theory involves the study of monogenic functions satisfying generalized Cauchy-Riemann or Riesz systems.

The present article presents the basics to discuss approximation properties for monogenic functions over 3D prolate spheroids by Fourier expansions in monogenic polynomials. The importance of constructing the underlying spheroidal monogenics stems from the role which they play in the calculation of the monogenic kernel functions. Once the kernel functions in a spheroid are known, it is possible to solve both basic boundary value and conformal mapping problems. In view of many applications to boundary value problems and for simplicity we shall base our discussion in

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[2]Technical University of Mining, Freiberg, Germany. E-mail: joao.pedro.morais@ua.pt

the approximation of functions defined in domains of $\mathbb{R}^3$ with values in the reduced quaternions (identified with $\mathbb{R}^3$). This class of functions coincides with the solutions of the well known Riesz system and shows more analogies to complex holomorphic functions than the more general class of quaternion-valued monogenic functions satisfying the Moisil-Théodoresco system. Unfortunately, such a structure is not closed under the quaternionic multiplication. To make sense of this, Section 3 studies the approximation of monogenic functions in the linear space of square integrable functions over $\mathbb{R}$. More intuitively still, in the Lamé [4] and Stokes systems [5] the operators of some boundary value problems are not $\mathbb{H}$-linear but are nevertheless efficiently treated by means of quaternionic analysis tools. Therefore, the consideration of a real-linear Hilbert space also has its own importance.

The paper is organized as follows. After presenting some definitions and basic properties of quaternionic analysis in Section 2, Section 3 presents a set of polynomial solutions of the Riesz system, which is complete and orthogonal over the interior of prolate spheroids. This will be done in the spaces of square integrable functions over $\mathbb{R}$. By the nature of the given approach, it is easily verified that a part of the theory carries over to arbitrary ellipsoids in the three-dimensional space. The representations of these polynomials are given explicitly, ready to be implemented on a computer. In addition, we show a corresponding orthogonality of the same polynomials over the surface of the spheroids with respect to a suitable weight function. The used methods also allow a generalization to spaces of square integrable quaternion-valued functions. Besides its obvious importance this case will not be discussed in the present article since we will focus the discussion on real Hilbert spaces for conciseness here. Further investigations will be reported in a forthcoming paper. To the best of our knowledge this is done here for the first time.

2 Notation and definitions

For all what follows we will work in $\mathbb{H}$, the skew field of quaternions. This means we can express each element $\mathbf{z} \in \mathbb{H}$ uniquely in the form $\mathbf{z} = z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}$, with real numbers z_i $(i = 0, 1, 2, 3)$, where the imaginary units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ stand for the elements of the basis of $\mathbb{H}$, subject to the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1; \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

As usual, the real vector space $\mathbb{R}^4$ may be embedded in $\mathbb{H}$ by identifying the element $z := (z_0, z_1, z_2, z_3) \in \mathbb{R}^4$ with $\mathbf{z} := z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k} \in \mathbb{H}$. In the sequel, consider the subset $\mathcal{A} := \text{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}\}$ of $\mathbb{H}$, then the real vector space $\mathbb{R}^3$ may be embedded in $\mathcal{A}$ via the identification of $x := (x_0, x_1, x_2) \in \mathbb{R}^3$ with the reduced quaternion $\mathbf{x} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} \in \mathcal{A}$. As a matter of fact, throughout the text we will often use the symbol x to represent a point in $\mathbb{R}^3$ and $\mathbf{x}$ to represent the corresponding reduced quaternion. Also, we emphasize that $\mathcal{A}$ is a real vectorial subspace, but not a subalgebra, of $\mathbb{H}$. The scalar and vector parts of $\mathbf{x}$, $\mathbf{Sc}(\mathbf{x})$ and $\mathbf{Vec}(\mathbf{x})$, are defined as the x_0 and $x_1\mathbf{i} + x_2\mathbf{j}$ terms, respectively. Like in the complex case, the conjugate of $\mathbf{x}$ is the reduced quaternion $\overline{\mathbf{x}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j}$, and the norm $|\mathbf{x}|$ of $\mathbf{x}$ is defined by $|\mathbf{x}|^2 = \mathbf{x}\overline{\mathbf{x}} = \overline{\mathbf{x}}\mathbf{x} = x_0^2 + x_1^2 + x_2^2$, which coincides with its corresponding Euclidean norm as a vector in $\mathbb{R}^3$. Now, let Ω be an open subset of $\mathbb{R}^3$ with a piecewise smooth boundary. We say that

$$\mathbf{f} : \Omega \longrightarrow \mathcal{A}, \quad \mathbf{f}(x) = [\mathbf{f}(x)]_0 + [\mathbf{f}(x)]_1\mathbf{i} + [\mathbf{f}(x)]_2\mathbf{j}$$

is a reduced quaternion-valued function or, briefly, an $\mathcal{A}$-valued function, where $[\mathbf{f}]_i$ $(i = 0, 1, 2)$ are real-valued functions defined in Ω. Properties such as continuity, differentiability, integrability, and so on, which are ascribed to $\mathbf{f}$ have to be fulfilled by all components $[\mathbf{f}]_i$. We further introduce the real-linear Hilbert space of square integrable $\mathcal{A}$-valued functions defined in Ω, that we denote by $L_2(\Omega;\mathcal{A};\mathbb{R})$. The scalar inner product is defined by

$$< \mathbf{f},\mathbf{g} >_{L_2(\Omega;\mathcal{A};\mathbb{R})} = \int_\Omega \mathbf{Sc}(\overline{\mathbf{f}}\,\mathbf{g})\,dV\,, \tag{1}$$

where dV denotes the Lebesgue measure in $\mathbb{R}^3$. To simplify matters further we shall remark that using the embedding of $\mathbb{R}$ in $\mathcal{A}$ the inner product of two real-valued functions $f,g:\Omega \longrightarrow \mathbb{R}$ can also be written by using the inner product (1), and it will be denoted simply by $< f,g >_{L_2(\Omega)}$.

Matters become interesting when we consider the notion of monogenicity, which is introduced by means of the so-called generalized Cauchy-Riemann operator

$$D = \frac{\partial}{\partial x_0} + \mathbf{i}\,\frac{\partial}{\partial x_1} + \mathbf{j}\,\frac{\partial}{\partial x_2}. \tag{2}$$

Definition 2.1 (Monogenicity) A continuously real-differentiable $\mathcal{A}$-valued function $\mathbf{f}$ is called monogenic in Ω if $D\mathbf{f}=0$ in Ω.

As the generalized Cauchy-Riemann operator (2) and its conjugate

$$\overline{D} = \frac{\partial}{\partial x_0} - \mathbf{i}\,\frac{\partial}{\partial x_1} - \mathbf{j}\,\frac{\partial}{\partial x_2} \tag{3}$$

factorize the Laplace operator in $\mathbb{R}^3$ in the sense that $\Delta_3 = D\overline{D} = \overline{D}D$, it follows that a monogenic function in Ω is harmonic in Ω, and so are all its components.

A big step towards is the realization that any monogenic $\mathcal{A}$-valued function is two-sided monogenic. This means that it satisfies simultaneously the equations $D\mathbf{f} = \mathbf{f}D = 0$, which are equivalent to the system

$$\text{(R)} \quad \left\{ \begin{array}{rcl} \operatorname{div}\overline{\mathbf{f}} & = & 0 \\ \operatorname{curl}\overline{\mathbf{f}} & = & 0 \end{array} \right. \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \partial_{x_0}[\mathbf{f}]_0 - \displaystyle\sum_{i=1}^{2} \partial_{x_i}[\mathbf{f}]_i = 0 \\ \qquad \partial_{x_j}[\mathbf{f}]_i + \partial_{x_i}[\mathbf{f}]_j = 0 \qquad (i \neq j,\ 0 \leq i,j \leq 2). \end{array} \right.$$

As is well known, the system (R) is called the Riesz system [10]. It clearly generalizes the classical Cauchy-Riemann system for holomorphic functions in the complex plane.

Following [8], the solutions of the system (R) are called (R)-solutions. The subspace of polynomial (R)-solutions of degree n will be denoted by $\mathcal{R}^+(\Omega;\mathcal{A};n)$. In [8], it is shown that the space $\mathcal{R}^+(\Omega;\mathcal{A};n)$ has dimension $2n+3$. We also denote by $\mathcal{R}^+(\Omega;\mathcal{A}) := L_2(\Omega;\mathcal{A};\mathbb{R}) \cap \ker D$ the space of square integrable $\mathcal{A}$-valued monogenic functions defined in Ω.

3 Results

In order to make it self-contained and to fix the notation we start by introducing cylindrical coordinates: $x_0 = z$, $x_1 = \rho\cos\phi$, $x_2 = \rho\sin\phi$, where $z \in (-\infty,+\infty)$, $\rho \in [0,+\infty)$ and $\phi \in [0,2\pi)$. Each point $x = (x_0,x_1,x_2) \in \mathbb{R}^3 \setminus \{0\}$ admits a unique representation $\mathbf{x} = z + \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j}$, where $|x| = \sqrt{z^2+\rho^2}$, emphasizing that $\phi = \arccos\left(\frac{x_1}{\rho}\right)$ if $\rho > 0$ and $\phi = 0$ if $\rho = 0$. In the present section we are interested in obtaining a complete orthogonal system of monogenic polynomials in the interior of the surface of revolution

$$\mathcal{E}:\ \frac{z^2}{\cosh^2\alpha} + \frac{\rho^2}{\sinh^2\alpha} = 1, \tag{4}$$

where α is a nonnegative real number. Representation (4) shows that surfaces of constant α do indeed form prolate spheroids, since they are ellipses rotated about the axis joining their foci.

Following [2] at this stage it is convenient to introduce the so-called elliptic-cylindrical coordinates, u and v, defined by the relations:

$$\begin{cases} z &= \cos u \cosh v \\ \rho &= \sin u \sinh v \end{cases},$$

where $u \in [0, \pi]$ and $v \in [0, a]$. In this case, the boundary of the above spheroid has the equation $v = \alpha$.

Partially inspired by the results from [9], we now consider a special system of monogenic polynomials with respect to the variables u, v, and the azimuthal angle ϕ. We will designate them by

$$\boldsymbol{\mathcal{E}}_{n,l},\ \boldsymbol{\mathcal{F}}_{n,m},\quad l = 0, \ldots, n+1,\ m = 1, \ldots, n+1$$

namely functions of the form

$$\begin{aligned} \boldsymbol{\mathcal{E}}_{n,l}\,(u, v, \phi) \;&:=\; \frac{(n+l+1)}{2} A_{n,l}(u, v)\, T_l(\cos\phi) \\ &+\; \frac{1}{4} A_{n,l+1}(u, v)\, [T_{l+1}(\cos\phi)\mathbf{i} + \sin\phi\, U_l(\cos\phi)\mathbf{j}] \\ &+\; \frac{1}{4} (n+1+l)(n+l) A_{n,l-1}(u, v)\, [-T_{l-1}(\cos\phi)\mathbf{i} + \sin\phi\, U_{l-2}(\cos\phi)\mathbf{j}] \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\mathcal{F}}_{n,m}\,(u, v, \phi) \;&:=\; \frac{(n+m+1)}{2} A_{n,m}(u, v) \sin\phi\, U_{m-1}(\cos\phi) \\ &+\; \frac{1}{4} A_{n,m+1}(u, v)\, [\sin\phi\, U_m(\cos\phi)\mathbf{i} - T_{m+1}(\cos\phi)\mathbf{j}] \\ &-\; \frac{1}{4} (n+1+m)(n+m) A_{n,m-1}(u, v)\, [\sin\phi\, U_{m-2}(\cos\phi)\mathbf{i} + T_{m-1}(\cos\phi)\mathbf{j}] \end{aligned}$$

with the notation

$$A_{n,l}(u, v) \;:=\; \sum_{k=0}^{\lceil \frac{n-l}{2} \rceil} \frac{(2n+1-2k)\,(n+l)_{2k}}{(n+1-l)_{2k+1}}\, P^l_{n-2k}(\cos u)\, P^l_{n-2k}(\cosh v).$$

Here, and throughout the present paper, $(a)_r$ denotes the Pochhammer symbol, i.e. $(a)_r := a(a+1)\cdots(a+r-1) = \frac{\Gamma(a+r)}{\Gamma(r)}$, for any integer $r > 1$, and $(a)_0 := 1$. As usual, $\lceil s \rceil$ denotes the smallest integer not less than $s \in \mathbb{R}$. In addition, P^l_n stands for the Ferrer's associated Legendre functions of degree n and order l of the first kind, T_l and U_l are the Chebyshev polynomials of the first and second kinds, respectively. For a more unified formulation we notice that for the case $l = 0$, the coefficient function $A_{n,-1}(u, v)$ is defined by $A_{n,-1} = -\frac{1}{n(n+1)} A_{n,1}$, and the Ferrer's associated Legendre function P^i_n is the zero function for $i \geq n+1$. Also, we set $P_n(\cosh v) = P^0_n(\cosh v)$ and $P^l_n(\cosh v) = (-1)^l (\sinh v)^l P^{(l)}_n(\cosh v)$, where $P^{(l)}_n(\cosh v) = \frac{d^l}{dt^l}[P_n(t)]\Big|_{t=\cosh v}$. One may show by elementary reasoning that the well-known recurrence properties of the Legendre polynomials and their associated Legendre functions are different from the ones we are familiar with.

Remark 3.1 *Although the monogenic polynomials $\boldsymbol{\mathcal{E}}_{n,0}$ involve Legendre polynomials while $\boldsymbol{\mathcal{E}}_{n,l}$ involve Ferrer's associated Legendre functions, we still include the treatment of the first into the general case, whenever this does not raise any confusion and the treatment remains the same. This separation becomes important if one needs to calculate the L_2-norms (over the surface of a spheroid) of $\boldsymbol{\mathcal{E}}_{n,0}$ and $\boldsymbol{\mathcal{E}}_{n,l}$, respectively.*

Remark 3.2 *We note that the harmonic polynomials* $\mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l})$ *and* $\mathbf{Sc}(\boldsymbol{\mathcal{F}}_{n,m})$ *are deliberately similar, up to a real constant depending on* l *or* m *and not only on the degree of the polynomial, to the ones exploited in [2]. However, the publication [2] was not focused on sets of monogenic polynomials.*

In the considerations to follow we will often omit the argument and write simply $\boldsymbol{\mathcal{E}}_{n,l}$ and $\boldsymbol{\mathcal{F}}_{n,m}$ instead of $\boldsymbol{\mathcal{E}}_{n,l}(u,v,\phi)$ and $\boldsymbol{\mathcal{F}}_{n,m}(u,v,\phi)$. Based on the previous representations we formulate a first preliminary result.

Lemma 3.1 *The monogenic polynomials* $\boldsymbol{\mathcal{E}}_{n,l}$ *and* $\boldsymbol{\mathcal{F}}_{n,m}$ *are the zero functions for* $l, m \geq n+2$.

Partly motivated by the results from [2], in the following we address the orthogonality of the above-mentioned polynomials over the interior of the prolate spheroid (4) in the sense of the scalar product (1), which is the main theme of the present section.

Theorem 3.1 *For each* $n \in \mathbb{N}_0$*, the set* $\{\boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} : l = 0,\ldots,n+1, m = 1,\ldots,n+1\}$ *is orthogonal over the interior of the prolate spheroid (4) in the sense of the scalar product (1).*

Proof: For a fixed n, we begin by proving the orthogonality of the monogenic polynomials $\boldsymbol{\mathcal{E}}_{n,l}$ $(l = 0,\ldots,n+1)$. By definition of the scalar product (1) for a fixed $n \in \mathbb{N}_0$ we have

$$\begin{aligned}
< \boldsymbol{\mathcal{E}}_{n,l_1}, \boldsymbol{\mathcal{E}}_{n,l_2} >_{L_2(\mathcal{E};\mathcal{A};\mathbb{R})} &= \int_{\mathcal{E}} \mathbf{Sc}(\overline{\boldsymbol{\mathcal{E}}_{n,l_1}}\, \boldsymbol{\mathcal{E}}_{n,l_2})\, dV \\
&= \underbrace{\int_{\mathcal{E}} \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l_1}) \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l_2})\, dV}_{\text{(I)}} + \underbrace{\int_{\mathcal{E}} ([\boldsymbol{\mathcal{E}}_{n,l_1}]_1 [\boldsymbol{\mathcal{E}}_{n,l_2}]_1 + [\boldsymbol{\mathcal{E}}_{n,l_1}]_2 [\boldsymbol{\mathcal{E}}_{n,l_2}]_2)\, dV}_{\text{(II)}}.
\end{aligned}$$

Assume we have the change of variables $z = z(u,v) = \cos u \cosh v$ and $\rho = \rho(u,v) = \sin u \sinh v$. A first straightforward computation shows

$$\begin{aligned}
\text{(I)} &= \int_{\mathcal{E}} \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l_1}) \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l_2})\, \rho\, \frac{\partial(u,v)}{\partial(z,\rho)}\, d\phi\, d\rho\, dz \\
&= \int_0^a \int_0^\pi A_{n,l_1}(u,v)\, A_{n,l_2}(u,v)\, \sin u\, \sinh v\, \frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \\
&\qquad \times \frac{(n+l_1+1)(n+l_2+1)}{4} \int_0^{2\pi} T_{l_1}(\cos\phi)\, T_{l_2}(\cos\phi)\, d\phi \\
&= 0, \quad l_1 \neq l_2.
\end{aligned}$$

Also, we can easily compute the remaining integral

$$\begin{aligned}
\text{(II)} &= \int_{\mathcal{E}} ([\boldsymbol{\mathcal{E}}_{n,l_1}]_1 [\boldsymbol{\mathcal{E}}_{n,l_2}]_1 + [\boldsymbol{\mathcal{E}}_{n,l_1}]_2 [\boldsymbol{\mathcal{E}}_{n,l_2}]_2)\, \rho\, \frac{\partial(u,v)}{\partial(z,\rho)}\, d\phi\, d\rho\, dz \\
&= \frac{1}{16} \int_0^a \int_0^\pi A_{n,l_1+1}(u,v)\, A_{n,l_2+1}(u,v)\, \sin u\, \sinh v\, \frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \\
&\times \int_0^{2\pi} [T_{l_1+1}(\cos\phi)\, T_{l_2+1}(\cos\phi) + \sin\phi\, U_{l_1}(\cos\phi)\, \sin\phi\, U_{l_2}(\cos\phi)]\, d\phi \\
&- \frac{1}{16}(n+1+l_1)(n+l_1) \int_0^a \int_0^\pi A_{n,l_1-1}(u,v)\, A_{n,l_2+1}(u,v)\, \sin u\, \sinh v \\
&\times \frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \int_0^{2\pi} [T_{l_1-1}(\cos\phi)\, T_{l_2+1}(\cos\phi) - \sin\phi\, U_{l_1-2}(\cos\phi)\, \sin\phi\, U_{l_2}(\cos\phi)]\, d\phi
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{16}(n+1+l_2)(n+l_2) \int_0^a \int_0^\pi A_{n,l_1+1}(u,v)\, A_{n,l_2-1}(u,v) \,\sin u \,\sinh v \\
&\times \frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \int_0^{2\pi} [T_{l_1+1}(\cos\phi)\, T_{l_2-1}(\cos\phi) - \sin\phi\, U_{l_1}(\cos\phi) \,\sin\phi\, U_{l_2-2}(\cos\phi)]\, d\phi \\
&+ \frac{1}{16}(n+1+l_1)(n+l_1)(n+1+l_2)(n+l_2) \int_0^a \int_0^\pi A_{n,l_1-1}(u,v) \\
&\times A_{n,l_2-1}(u,v) \,\sin u \,\sinh v \,\frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \int_0^{2\pi} [T_{l_1-1}(\cos\phi)\, T_{l_2-1}(\cos\phi) \\
&\qquad + \sin\phi\, U_{l_1-2}(\cos\phi) \,\sin\phi\, U_{l_2-2}(\cos\phi)]\, d\phi.
\end{aligned}
$$

We remark that some of the previous integrals with the terms $l_1 \neq l_2$ vanish because of the orthogonality of the Chebyshev polynomials. In this sense, we consider now only if $l_1 = l_2 + 2$ (analogues $l_1 = l_2 - 2$), and the previous expression reduces to the form

$$
\begin{aligned}
&\int_{\mathcal{E}} ([\boldsymbol{\mathcal{E}}_{n,l_1}]_1 [\boldsymbol{\mathcal{E}}_{n,l_2}]_1 + [\boldsymbol{\mathcal{E}}_{n,l_1}]_2 [\boldsymbol{\mathcal{E}}_{n,l_2}]_2)\ \rho\, \frac{\partial(u,v)}{\partial(z,\rho)}\, d\phi\, d\rho\, dz \\
&= -\frac{1}{16}\,(n+3+l_2)(n+2+l_2) \int_0^a \int_0^\pi A_{n,l_2+1}(u,v)\, A_{n,l_2+1}(u,v) \\
&\times \sin u \,\sinh v \,\frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \int_0^{2\pi} [T_{l_2+1}(\cos\phi)\, T_{l_2+1}(\cos\phi) - \sin\phi\, U_{l_2}(\cos\phi) \,\sin\phi\, U_{l_2}(\cos\phi)]\, d\phi \\
&= 0, \quad l_2 = 0, \ldots, n+1.
\end{aligned}
$$

Therefore the (II)-integral vanishes for $l_1 \neq l_2$, and consequently $< \boldsymbol{\mathcal{E}}_{n,l_1}, \boldsymbol{\mathcal{E}}_{n,l_2} >_{L_2(\mathcal{E};\mathcal{A};\mathbb{R})} = 0$ for $l_1 \neq l_2$. In an analogous way, for the polynomials $\boldsymbol{\mathcal{F}}_{n,m}$ $(m = 1, \ldots, n+1)$, we conclude that $< \boldsymbol{\mathcal{F}}_{n,m_1}, \boldsymbol{\mathcal{F}}_{n,m_2} >_{L_2(\mathcal{E};\mathcal{A};\mathbb{R})} = 0$ for $m_1 \neq m_2$. Now, for a fixed $n \in \mathbb{N}_0$ note that

$$
\begin{aligned}
< \boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} >_{L_2(\mathcal{E};\mathcal{A};\mathbb{R})} &= \int_{\mathcal{E}} \mathbf{Sc}(\overline{\boldsymbol{\mathcal{E}}_{n,l}}\, \boldsymbol{\mathcal{F}}_{n,m})\, dV \\
&= \underbrace{\int_{\mathcal{E}} \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l}) \mathbf{Sc}(\boldsymbol{\mathcal{F}}_{n,m})\, dV}_{\text{(III)}} + \underbrace{\int_{\mathcal{E}} ([\boldsymbol{\mathcal{E}}_{n,l}]_1 [\boldsymbol{\mathcal{F}}_{n,m}]_1 + [\boldsymbol{\mathcal{E}}_{n,l}]_2 [\boldsymbol{\mathcal{F}}_{n,m}]_2)\, dV}_{\text{(IV)}}.
\end{aligned}
$$

A first straightforward computation shows

$$
\begin{aligned}
\text{(III)} &= \int_{\mathcal{E}} \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l}) \mathbf{Sc}(\boldsymbol{\mathcal{F}}_{n,m})\, \rho\, \frac{\partial(u,v)}{\partial(z,\rho)}\, d\phi\, d\rho\, dz \\
&= \int_0^a \int_0^\pi A_{n,l}(u,v)\, A_{n,m}(u,v) \,\sin u \sinh v \,\frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \\
&\qquad \times \frac{(n+l+1)(n+m+1)}{4} \int_0^{2\pi} T_l(\cos\phi) \,\sin\phi\, U_{m-1}(\cos\phi)\, d\phi \\
&= 0, \quad l = 0, \ldots, n+1, \ m = 1, \ldots, n+1.
\end{aligned}
$$

The remaining integral can be easily computed

$$
\begin{aligned}
\text{(IV)} \;=\; & \int_{\mathcal{E}} ([\boldsymbol{\mathcal{E}}_{n,l}]_1[\boldsymbol{\mathcal{F}}_{n,m}]_1 + [\boldsymbol{\mathcal{E}}_{n,l}]_2[\boldsymbol{\mathcal{F}}_{n,m}]_2)\; \rho\, \frac{\partial(u,v)}{\partial(z,\rho)}\, d\phi\, d\rho\, dz \\
= \;& \frac{1}{16}\int_0^a \int_0^\pi A_{n,l+1}(u,v)\, A_{n,m+1}(u,v)\, \sin u\, \sinh v\, \frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \\
\times \;& \int_0^{2\pi} [T_{l+1}(\cos\phi)\, \sin\phi\, U_m(\cos\phi) - \sin\phi\, U_l(\cos\phi)\, T_{m+1}(\cos\phi)]\, d\phi \\
- \;& \frac{1}{16}(n+1+m)(n+m)\int_0^a \int_0^\pi A_{n,l+1}(u,v)\, A_{n,m-1}(u,v)\, \sin u\, \sinh v \\
\times \;& \frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \int_0^{2\pi} [T_{l+1}(\cos\phi)\, \sin\phi\, U_{m-2}(\cos\phi) + \sin\phi\, U_l(\cos\phi)\, T_{m-1}(\cos\phi)]\, d\phi \\
- \;& \frac{1}{16}(n+1+l)(n+l)\int_0^a \int_0^\pi A_{n,l-1}(u,v)\, A_{n,m+1}(u,v)\, \sin u\, \sinh v \\
\times \;& \frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \int_0^{2\pi} [T_{l-1}(\cos\phi)\, \sin\phi\, U_m(\cos\phi) - \sin\phi\, U_{l-2}(\cos\phi)\, T_{m+1}(\cos\phi)]\, d\phi \\
- \;& \frac{1}{16}(n+1+l)(n+l)(n+1+m)(n+m)\int_0^a \int_0^\pi A_{n,l-1}(u,v) \\
\times \;& A_{n,m-1}(u,v)\, \sin u\, \sinh v\, \frac{\partial(u,v)}{\partial(z,\rho)}\, du\, dv \int_0^{2\pi} [T_{l-1}(\cos\phi)\sin\phi\, U_{m-2}(\cos\phi) \\
& \qquad - \sin\phi\, U_{l-2}(\cos\phi)\, T_{m-1}(\cos\phi)]\, d\phi.
\end{aligned}
$$

Using the orthogonal properties of the Chebyshev polynomials and by the same reasoning as before, we may conclude that (IV)-integral vanishes, and consequently $< \boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} >_{L_2(\mathcal{E};\mathcal{A};\mathbb{R})} = 0$ for $l = 0, \ldots, n+1$ and $m = 1, \ldots, n+1$. Thus, for a fixed $n \in \mathbb{N}_0$ the set $\{\boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} : l = 0, \ldots, n+1, m = 1, \ldots, n+1\}$ is orthogonal in the sense of the scalar product (1). We study now the orthogonality of the system for each degree n. The proof is mainly based on the expressions of the aforementioned spheroidal monogenics combining several recurrence properties of the Legendre polynomials and their associated Legendre functions, and some equations between the Chebyshev polynomials of the first and second kinds. Therefore, it requires extensive calculations. Here, we only give the main steps of the proof. We start with the monogenic polynomials $\boldsymbol{\mathcal{E}}_{n,l}$ $(l = 0, \ldots, n+1)$. By definition of the scalar product (1) it follows

$$
< \boldsymbol{\mathcal{E}}_{n_1,l}, \boldsymbol{\mathcal{E}}_{n_2,l} >_{L_2(\mathcal{E};\mathcal{A};\mathbb{R})} = \underbrace{\int_{\mathcal{E}} \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n_1,l})\mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n_2,l})\, dV}_{\text{(V)}} + \underbrace{\int_{\mathcal{E}} ([\boldsymbol{\mathcal{E}}_{n_1,l}]_1[\boldsymbol{\mathcal{E}}_{n_2,l}]_1 + [\boldsymbol{\mathcal{E}}_{n_1,l}]_2[\boldsymbol{\mathcal{E}}_{n_2,l}]_2)\, dV}_{\text{(VI)}}.
$$

In the following we start by presenting the calculations for the integral (V). The calculations for (VI) follow the same principle and are therefore straightforward. In the sequel, let h be a harmonic polynomial of variables u, v, and ϕ. Analogously to the results in [2], for each $n \in \mathbb{N}_0$ a direct computation shows that

$$
\begin{aligned}
< \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l}), h >_{L_2(\mathcal{E})} &= \int_{\mathcal{E}} \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l})\, h\, \rho\, \frac{\partial(u,v)}{\partial(z,\rho)}\, d\phi\, d\rho\, dz \\
&= \frac{(n+l+1)}{2}\int_{\mathcal{E}} A_{n,l}(u,v)\, T_l(\cos\phi)\, h\, \rho\, \frac{\partial(u,v)}{\partial(z,\rho)}\, d\phi\, d\rho\, dz. \qquad (5)
\end{aligned}
$$

By direct inspection of previous expressions a first straightforward computation shows

$$A_{n,l}(u,v) = \frac{1}{(n+1-l)}\left[(2n+1)P_n^l(\cos u)P_n^l(\cosh v) + (n+l)A\right],$$

where

$$\begin{aligned} A &= \frac{(n-1+l)}{(n-l)}\, A_{n-2,l}(u,v) \\ &= \frac{(n-1+l)}{(n-l)}\, \frac{1}{\sin^2 u + \sinh^2 v}\, \big[\cosh v\, P_{n-2}^l(\cos u)P_{n-1}^l(\cosh v) \\ &\qquad - \cos u\, P_{n-1}^l(\cos u)P_{n-2}^l(\cosh v)\big] . \end{aligned}$$

Now, making the change of variables $\cos u = t$ and $\cosh v = t$ in the previous expression, and using the recurrence formula

$$(n+1-l)P_{n+1}^l(t) - (2n+1)tP_n^l(t) + (n+l)P_{n-1}^l(t) = 0,$$

$l = 0, \ldots, n+1$, it follows that

$$\begin{aligned} A &= \frac{1}{\sin^2 u + \sinh^2 v}\left\{\cosh v\, P_{n-1}^l(\cosh v)\left[\frac{(2n-1)}{(n-l)}\, \cos u\, P_{n-1}^l(\cos u) - \frac{(n-1+l)}{(n-l)}\, P_{n-2}^l(\cos u)\right]\right. \\ &- \left.\cos u\, P_{n-1}^l(\cos u)\left[\frac{(2n-1)}{(n-l)}\, \cosh v\, P_{n-1}^l(\cosh v) - \frac{(n-1+l)}{(n-l)}\, P_{n-2}^l(\cosh v)\right]\right\}. \end{aligned}$$

With these calculations at hand, we set

$$\begin{aligned} A_{n,l}(u,v) &= \frac{(2n+1)}{(n+1-l)}\, P_n^l(\cos u)\, P_n^l(\cosh v) \\ &- \frac{(n+l)}{(n+1-l)}\, \frac{1}{\sin^2 u + \sinh^2 v}\, \big[\cosh v\, P_n^l(\cos u)\, P_{n-1}^l(\cosh v) \\ &\qquad - \cos u\, P_{n-1}^l(\cos u)\, P_n^l(\cosh v)\big] \\ &= \frac{1}{\sin^2 u + \sinh^2 v}\left\{P_n^l(\cos u)\, P_n^l(\cosh v)\left[\frac{(2n+1)}{(n+1-l)}\, \cosh^2 v - \frac{(2n+1)}{(n+1-l)}\, \cos^2 u\right]\right. \\ &\qquad - \frac{(n+l)}{(n+1-l)}\, P_n^l(\cos u)\, P_{n-1}^l(\cosh v)\, \cosh v \\ &\qquad \left. + \frac{(n+l)}{(n+1-l)}\, P_{n-1}^l(\cos u)\, P_n^l(\cosh v)\, \cos u\right\} \\ &= \frac{1}{\sin^2 u + \sinh^2 v}\left[\cosh v\, P_n^l(\cos u)\, P_{n+1}^l(\cosh v) - \cos u\, P_{n+1}^l(\cos u)\, P_n^l(\cosh v)\right]. \end{aligned}$$

Hence, substituting in (5) we finally obtain

$$\begin{aligned} < \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l}), h >_{L_2(\mathcal{E})} &= \frac{(n+l+1)}{2} \int_0^a \int_0^\pi \int_0^{2\pi} h\, T_l(\cos\phi)\, \sin u \sinh v \\ &\times \left[\cosh v\, P_n^l(\cos u)\, P_{n+1}^l(\cosh v) - \cos u\, P_{n+1}^l(\cos u)\, P_n^l(\cosh v)\right] d\phi\, du\, dv. \end{aligned}$$

The same value is obtained if we replace $T_l(\cos\phi)$ by $\sin\phi\, U_{m-1}(\cos\phi)$ for $m = 1, \dots, n+1$. Using similar ideas to those in [2], we may see that the last integral vanishes when h is a harmonic polynomial of the form

$$P_s^l(\cos u)\, P_s^l(\cosh v)\, T_l(\cos\phi)$$

with $s < n$, since

$$\int_0^\pi P_n^l(\cos u)\, \sin u\, P_s^l(\cos u)\, du \;=\; 0$$
$$\int_0^\pi P_{n+1}^l(\cos u)\, \cos u\, P_s^l(\cos u) \sin u\, du \;=\; 0.$$

Hence, for $n_1 \neq n_2$

$$\text{(V)} \;=< \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n_1,l}), \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n_2,l}) >_{L_2(\mathcal{E})} =\; 0,$$

and similarly

$$< \mathbf{Sc}(\boldsymbol{\mathcal{F}}_{n_1,m}), \mathbf{Sc}(\boldsymbol{\mathcal{F}}_{n_2,m}) >_{L_2(\mathcal{E})} =\; 0.$$

As we may now show, the remaining integrals

$$\text{(VI)} \;=\; \int_{\mathcal{E}} ([\boldsymbol{\mathcal{E}}_{n_1,l}]_1 [\boldsymbol{\mathcal{E}}_{n_2,l}]_1 + [\boldsymbol{\mathcal{E}}_{n_1,l}]_2 [\boldsymbol{\mathcal{E}}_{n_2,l}]_2)\, dV$$

and

$$\int_{\mathcal{E}} ([\boldsymbol{\mathcal{F}}_{n_1,m}]_1 [\boldsymbol{\mathcal{F}}_{n_2,m}]_1 + [\boldsymbol{\mathcal{F}}_{n_1,m}]_2 [\boldsymbol{\mathcal{F}}_{n_2,m}]_2)\, dV$$

are derived by adapting the previous arguments in a straightforward way, and they consist of very lengthy calculations combining several recurrence properties of the Legendre polynomials and their associated Legendre functions, and some identities between the Chebyshev polynomials of the first and second kinds. For each $n \in \mathbb{N}_0$, a few straightforward computations show that

$$\begin{aligned} < [\boldsymbol{\mathcal{E}}_{n,l}]_1, h >_{L_2(\mathcal{E})} \;=\;& \frac{1}{4} \int_0^a \int_0^\pi \int_0^{2\pi} h\, T_{l+1}(\cos\phi)\, \sin u \sinh v \\ \times\;& \left[\cosh v\, P_n^{l+1}(\cos u)\, P_{n+1}^{l+1}(\cosh v) - \cos u\, P_{n+1}^{l+1}(\cos u)\, P_n^{l+1}(\cosh v)\right] d\phi\, du\, dv \\ -\;& \frac{1}{4}(n+1+l)(n+l) \int_0^a \int_0^\pi \int_0^{2\pi} h\, T_{l-1}(\cos\phi)\, \sin u \sinh v \\ \times\;& \left[\cosh v\, P_n^{l-1}(\cos u)\, P_{n+1}^{l-1}(\cosh v) - \cos u\, P_{n+1}^{l-1}(\cos u)\, P_n^{l-1}(\cosh v)\right] d\phi\, du\, dv \end{aligned}$$

and

$$\begin{aligned} < [\boldsymbol{\mathcal{E}}_{n,l}]_2, h >_{L_2(\mathcal{E})} \;=\;& \frac{1}{4} \int_0^a \int_0^\pi \int_0^{2\pi} h\, \sin\phi\, U_l(\cos\phi)\, \sin u \sinh v \\ \times\;& \left[\cosh v\, P_n^{l+1}(\cos u)\, P_{n+1}^{l+1}(\cosh v) - \cos u\, P_{n+1}^{l+1}(\cos u)\, P_n^{l+1}(\cosh v)\right] d\phi\, du\, dv \\ +\;& \frac{1}{4}(n+1+l)(n+l) \int_0^a \int_0^\pi \int_0^{2\pi} h\, \sin\phi\, U_{l-2}(\cos\phi)\, \sin u \sinh v \\ \times\;& \left[\cosh v\, P_n^{l-1}(\cos u)\, P_{n+1}^{l-1}(\cosh v) - \cos u\, P_{n+1}^{l-1}(\cos u)\, P_n^{l-1}(\cosh v)\right] d\phi\, du\, dv. \end{aligned}$$

Now, using previous arguments and after some extensive calculations, we may show that the integral (VI) vanishes when h is respectively a harmonic polynomial of the forms

$$P_s^{l+1}(\cos u)\, P_s^{l+1}(\cosh v)\, T_{l+1}(\cos\phi) + P_s^{l-1}(\cos u)\, P_s^{l-1}(\cosh v)\, T_{l-1}(\cos\phi)$$

and

$$P_s^{l+1}(\cos u)\, P_s^{l+1}(\cosh v)\, \sin\phi\, U_l(\cos\phi) + P_s^{l-1}(\cos u)\, P_s^{l-1}(\cosh v)\, \sin\phi\, U_{l-2}(\cos\phi)$$

with $s < n$, since

$$\begin{aligned}
\int_0^\pi P_{n+1}^{l+1}(\cos u)\, \sin u\, \cos u\, P_s^{l+1}(\cos u)\, du &= 0\\
\int_0^\pi P_n^{l+1}(\cos u)\, \sin u\, P_s^{l+1}(\cos u)\, du &= 0\\
\int_0^\pi P_n^{l-1}(\cos u)\, \sin u\, P_s^{l+1}(\cos u)\, du &= 0\\
\int_0^\pi P_n^{l-1}(\cos u)\, \sin u\, P_s^{l-1}(\cos u)\, du &= 0\\
\int_0^\pi P_{n+1}^{l-1}(\cos u)\, \sin u\, \cos u\, P_s^{l+1}(\cos u)\, du &= 0\\
\int_0^\pi P_{n+1}^{l-1}(\cos u)\, \sin u\, \cos u\, P_s^{l-1}(\cos u)\, du &= 0,
\end{aligned}$$

and moreover,

$$\int_0^\pi \left[P_n^{l+1}(\cos u)\, \sin u\, P_s^{l-1}(\cos u) - P_{n+1}^{l+1}(\cos u)\, \sin u\, \cos u\, P_s^{l-1}(\cos u)\right] du = 0$$

with $s < n$ and $l > 0$. For $l = 0$ it is easy to see that

$$\begin{aligned}
\int_0^\pi P_n^1(\cos u)\, \sin u\, P_s^{-1}(\cos u)\, du &= 0\\
\int_0^\pi P_{n+1}^1(\cos u)\, \sin u\, \cos u\, P_s^{-1}(\cos u) du &= 0
\end{aligned}$$

with $s < n$. Hence

$$\text{(VI)} \;=\; \int_{\mathcal{E}} ([\boldsymbol{\mathcal{E}}_{n_1,l}]_1[\boldsymbol{\mathcal{E}}_{n_2,l}]_1 + [\boldsymbol{\mathcal{E}}_{n_1,l}]_2[\boldsymbol{\mathcal{E}}_{n_2,l}]_2)\, dV \;=\; 0,$$

and consequently, $< \boldsymbol{\mathcal{E}}_{n_1,l}, \boldsymbol{\mathcal{E}}_{n_2,l} >_{L_2(\mathcal{E};\mathcal{A};\mathbb{R})} = 0$ for $l \geq 0$. Similarly, one may show that $< \boldsymbol{\mathcal{F}}_{n_1,m}, \boldsymbol{\mathcal{F}}_{n_2,m} >_{L_2(\mathcal{E};\mathcal{A};\mathbb{R})} = 0$ for $m > 0$. In summary, the set $\{\boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} : l = 0, \ldots, n+1, m = 1, \ldots, n+1; n = 0, 1, \ldots\}$ is orthogonal in the sense of the scalar product (1).

□

Based on the results from [2] we may now prove the orthogonality of the set $\{\boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} : l = 0, \ldots, n+1, m = 1, \ldots, n+1; n = 0, 1, \ldots\}$ over the interior of the ellipse (4) for all values of α to obtain a corresponding orthogonality of the same system over the surface $\mathcal{S}$ of the spheroid $\mathcal{E}$ with respect to a suitable weight function. Next we formulate the result.

Theorem 3.2 *For each* $n \in \mathbb{N}_0$*, the set* $\{\boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} : l = 0, \ldots, n+1, m = 1, \ldots, n+1\}$ *is orthogonal over the surface of the spheroid (4) in the sense of the scalar product*

$$< \mathbf{f}, \mathbf{g} >_{L_2(\mathcal{S};\mathcal{A};\mathbb{R})} = \int_{\mathcal{S}} \mathbf{Sc}(\overline{\mathbf{f}}\, \mathbf{g})\, |1 - (z + \boldsymbol{i}\,\rho)^2|^{1/2}\, d\sigma\,,$$

with weight function $|1 - (z + \boldsymbol{i}\,\rho)^2|^{1/2}$ *equal to the square root of the product of the distances from* (z, ρ, ϕ) *to the points* $(1, 0, 0)$ *and* $(-1, 0, 0)$*. Here,* $d\sigma$ *denotes the Lebesgue measure on* $\mathcal{S}$*.*

Proof: For simplicity we only give the main steps of the proof. We start by presenting the calculations for the scalar parts of the monogenic polynomials $\boldsymbol{\mathcal{E}}_{n,l}$ $(l = 0, \ldots, n+1)$. Although the proof for the remaining coordinates requires extensive calculations, it follows the same principle and is therefore straightforward. For $s = n$, we have seen that

$$\begin{aligned} h &:= \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l}) \\ &= \frac{(n+1+l)}{2} \frac{(2n+1)}{(n+1-l)} P_n^l(\cos u)\, P_n^l(\cosh v)\, T_l(\cos\phi) \\ &+ \sum_{k=1}^{\lceil \frac{n-l}{2} \rceil} \frac{(n+1+l)\,(2n+1-2k)\,(n+l)_{2k}}{2\,(n+1-l)_{2k+1}} P_{n-2k}^l(\cos u)\, P_{n-2k}^l(\cosh v)\, T_l(\cos\phi) \end{aligned}$$

where the last summand indicates harmonic polynomials of lower degree, which are orthogonal to $P_n^l(\cos u)\, P_n^l(\cosh v)\, T_l(\cos\phi)$ (see previous theorem). Thus, by direct inspection of a previous expression a first straightforward computation shows that

$$\begin{aligned} & < \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n_1,l_1}), \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n_2,l_2}) >_{L_2(\mathcal{E})} \\ = & \frac{(n_1+1+l_1)^2}{4} \delta_{n_1,n_2}\, \delta_{l_1,l_2} \int_0^a \int_0^\pi \int_0^{2\pi} A_{n_1,l_1}(u,v)\, [T_{l_1}(\cos\phi)]^2 \sin u \sinh v \\ \times & \left[\cosh v\, P_{n_1}^{l_1}(\cos u)\, P_{n_1+1}^{l_1}(\cosh v) - \cos u\, P_{n_1+1}^{l_1}(\cos u)\, P_{n_1}^{l_1}(\cosh v)\right] d\phi\, du\, dv \\ = & \frac{(n_1+1+l_1)}{2} \frac{(n_1+1+l_1)!}{(n_1+1-l_1)!} (1+\delta_{0,l_1})\pi\, \delta_{n_1,n_2}\, \delta_{l_1,l_2} \;\times \\ \times & \left[\int_0^a P_{n_1}^{l_1}(\cosh v)\, \sinh v\, \cosh v\, P_{n_1+1}^{l_1}(\cosh v)\, dv \right. \\ & \left. - \frac{(n_1+1+l_1)}{2n_1+3} \int_0^a P_{n_1}^{l_1}(\cosh v)\, \sinh v\, P_{n_1}^{l_1}(\cosh v)\, dv \right]. \end{aligned}$$

Let α be a nonnegative real number. By the same reasoning as in [2], it follows

$$\begin{aligned} \frac{d}{d\alpha} < \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n_1,l_1}), \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n_2,l_2}) >_{L_2(\mathcal{E})} &= \frac{(n_1+1+l_1)}{2} \frac{(n_1+1+l_1)!}{(n_1+1-l_1)!} (1+\delta_{0,l_1})\pi\, \delta_{n_1,n_2}\, \delta_{l_1,l_2} \\ &\times \left[\frac{d}{d\alpha} \int_0^a P_{n_1}^{l_1}(\cosh v)\, \sinh v\, \cosh v\, P_{n_1+1}^{l_1}(\cosh v)\, dv \right. \\ &\left. - \frac{(n_1+1+l_1)}{2n_1+3} \frac{d}{d\alpha} \int_0^a [P_{n_1}^{l_1}(\cosh v)]^2\, \sinh v\, dv \right], \end{aligned}$$

whence

$$\int_{\mathcal{S}} \mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n_1,l_1})\,\mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n_2,l_2})\,|1-(z+\mathbf{i}\,\rho)^2|^{1/2}\,d\sigma$$

$$= \frac{(n_1+1+l_1)}{2}\,\frac{(n_1+1+l_1)!}{(n_1+1-l_1)!}\Big[P^{l_1}_{n_1}(\cosh\alpha)\,\sinh\alpha\,\cosh\alpha\,P^{l_1}_{n_1+1}(\cosh\alpha)$$
$$- \frac{(n_1+1+l_1)}{2n_1+3}\,[P^{l_1}_{n_1}(\cosh\alpha)]^2\,\sinh\alpha\Big]\,\delta_{n_1,n_2}\,\delta_{l_1,l_2}\,(1+\delta_{0,l_1})\pi\,,$$

with exactly the same formulas in both cases if $T_l(\cos\phi)$ is replaced by $\sin\phi\,U_{m-1}(\cos\phi)$, $m>0$. Here the weight function $|1-(z+\mathbf{i}\,\rho)^2|^{1/2}$ equals the square root of the product of the distances from (z,ρ,ϕ) to the north and south poles, $(1,0,0)$ and $(-1,0,0)$, respectively. In an analogous way, for the remaining integrals we adapt the previous arguments after some extensive calculations while combining the following identities (for $s=n$):

$$\begin{aligned}
h &:= [\boldsymbol{\mathcal{E}}_{n,l}]_1\\
&= \frac{1}{4}\,\frac{(2n+1)}{(n-l)}\,P^{l+1}_n(\cos u)\,P^{l+1}_n(\cosh v)\,T_{l+1}(\cos\phi)\\
&- \frac{1}{4}\,(n+1+l)\,(n+l)\,\frac{(2n+1)}{(n+2-l)}\,P^{l-1}_n(\cos u)\,P^{l-1}_n(\cosh v)\,T_{l-1}(\cos\phi)\\
&+ \sum_{k=1}^{\lceil\frac{n-1-l}{2}\rceil}\frac{(2n+1-2k)\,(n+1+l)_{2k}}{4\,(n-l)_{2k+1}}\,P^{l+1}_{n-2k}(\cos u)\,P^{l+1}_{n-2k}(\cosh v)\,T_{l+1}(\cos\phi)\\
&- \sum_{k=1}^{\lceil\frac{n+1-l}{2}\rceil}\frac{(n+1+l)\,(n+l)\,(2n+1-2k)\,(n-1+l)_{2k}}{4\,(n+2-l)_{2k+1}}\,P^{l-1}_{n-2k}(\cos u)P^{l-1}_{n-2k}(\cosh v)\,T_{l-1}(\cos\phi)
\end{aligned}$$

and

$$\begin{aligned}
h &:= [\boldsymbol{\mathcal{E}}_{n,l}]_2\\
&= \frac{1}{4}\,\frac{(2n+1)}{(n-l)}\,P^{l+1}_n(\cos u)\,P^{l+1}_n(\cosh v)\,\sin\phi\,U_l(\cos\phi)\\
&+ \frac{1}{4}\,(n+1+l)\,(n+l)\,\frac{(2n+1)}{(n+2-l)}\,P^{l-1}_n(\cos u)\,P^{l-1}_n(\cosh v)\,\sin\phi\,U_{l-2}(\cos\phi)\\
&+ \sum_{k=1}^{\lceil\frac{n-1-l}{2}\rceil}\frac{(2n+1-2k)\,(n+1+l)_{2k}}{4\,(n-l)_{2k+1}}\,P^{l+1}_{n-2k}(\cos u)P^{l+1}_{n-2k}(\cosh v)\,\sin\phi\,U_l(\cos\phi)\\
&+ \sum_{k=1}^{\lceil\frac{n+1-l}{2}\rceil}\frac{(n+1+l)\,(n+l)\,(2n+1-2k)\,(n-1+l)_{2k}}{4\,(n+2-l)_{2k+1}}\,P^{l-1}_{n-2k}(\cos u)\;\times\\
&\qquad\times\,P^{l-1}_{n-2k}(\cosh v)\,\sin\phi\,U_{l-2}(\cos\phi)
\end{aligned}$$

where the last two summands in each expression indicate harmonic polynomials of lower degree, which are, respectively, orthogonal to (see previous theorem)

$$P^{l+1}_n(\cos u)\,P^{l+1}_n(\cosh v)\,T_{l+1}(\cos\phi)+P^{l-1}_n(\cos u)\,P^{l-1}_n(\cosh v)\,T_{l-1}(\cos\phi)$$

and

$$P_n^{l+1}(\cos u)\,P_n^{l+1}(\cosh v)\,\sin\phi\,U_l(\cos\phi)+P_n^{l-1}(\cos u)\,P_n^{l-1}(\cosh v)\,\sin\phi\,U_{l-2}(\cos\phi).$$

Thus, by direct inspection of the previous expressions a first straightforward computation shows that

$$\begin{aligned}
&< [\boldsymbol{\mathcal{E}}_{n_1,l_1}]_1, [\boldsymbol{\mathcal{E}}_{n_2,l_2}]_1 >_{L_2(\mathcal{E})} + < [\boldsymbol{\mathcal{E}}_{n_1,l_1}]_2, [\boldsymbol{\mathcal{E}}_{n_2,l_2}]_2 >_{L_2(\mathcal{E})} \\
= \;& \frac{1}{4}\,\frac{1}{(n_1-l_1)}\,\pi\,\delta_{n_1,n_2}\,\delta_{l_1,l_2}\left[\frac{(n_1+1+l_1)!}{(n_1-1-l_1)!}\int_0^a P_{n_1}^{l_1+1}(\cosh v)\,\sinh v\,\cosh v\,P_{n_1+1}^{l_1+1}(\cosh v)\,dv\right. \\
&\left. -\;\frac{1}{2n_1+3}\,\frac{(n_1+2+l_1)!}{(n_1-1-l_1)!}\int_0^a [P_{n_1}^{l_1+1}(\cosh v)]^2\,\sinh v\,dv\right] \\
+ \;& \frac{1}{4}\,\frac{(n_1+1+l_1)^2\,(n+l_1)^2}{(n_1+2-l_1)}\,(1-\delta_{0,l_1})\pi\,\delta_{n_1,n_2}\,\delta_{l_1,l_2} \\
\times \;& \left[\frac{(n_1-1+l_1)!}{(n_1+1-l_1)!}\int_0^a P_{n_1}^{l_1-1}(\cosh v)\,\sinh v\,\cosh v\,P_{n_1+1}^{l_1-1}(\cosh v)\,dv\right. \\
&\left. -\;\frac{1}{2n_1+3}\,\frac{(n_1+l_1)!}{(n_1+1-l_1)!}\int_0^a [P_{n_1}^{l_1-1}(\cosh v)]^2\,\sinh v\,dv\right].
\end{aligned}$$

By the same reasoning as before, it follows

$$\begin{aligned}
&\frac{d}{d\alpha}\left[< [\boldsymbol{\mathcal{E}}_{n_1,l_1}]_1, [\boldsymbol{\mathcal{E}}_{n_2,l_2}]_1 >_{L_2(\mathcal{E})} + < [\boldsymbol{\mathcal{E}}_{n_1,l_1}]_2, [\boldsymbol{\mathcal{E}}_{n_2,l_2}]_2 >_{L_2(\mathcal{E})}\right] \\
= \;& \frac{1}{4}\,\frac{1}{(n_1-l_1)}\,\frac{(n_1+1+l_1)!}{(n_1-1-l_1)!}\,\pi\,\delta_{n_1,n_2}\,\delta_{l_1,l_2}\left[\frac{d}{d\alpha}\int_0^a P_{n_1}^{l_1+1}(\cosh v)\,\sinh v\,\cosh v\,P_{n_1+1}^{l_1+1}(\cosh v)\,dv\right. \\
&\left. -\;\frac{(n_1+2+l_1)}{2n_1+3}\,\frac{d}{d\alpha}\int_0^a [P_{n_1}^{l_1+1}(\cosh v)]^2\,\sinh v\,dv\right] \\
+ \;& \frac{1}{4}\,\frac{(n_1+1+l_1)^2\,(n+l_1)^2}{(n_1+2-l_1)}\,\frac{(n_1-1+l_1)!}{(n_1+1-l_1)!}\,(1-\delta_{0,l_1})\pi\,\delta_{n_1,n_2}\,\delta_{l_1,l_2} \\
\times \;& \left[\frac{d}{d\alpha}\int_0^a P_{n_1}^{l_1-1}(\cosh v)\,\sinh v\,\cosh v\,P_{n_1+1}^{l_1-1}(\cosh v)\,dv\right. \\
&\left. -\;\frac{(n_1+l_1)}{2n_1+3}\,\frac{d}{d\alpha}\int_0^a [P_{n_1}^{l_1-1}(\cosh v)]^2\,\sinh v\,dv\right],
\end{aligned}$$

whence

$$\begin{aligned}
&\int_{\mathcal{S}}\left([\boldsymbol{\mathcal{E}}_{n_1,l_1}]_1\,[\boldsymbol{\mathcal{E}}_{n_2,l_2}]_1+[\boldsymbol{\mathcal{E}}_{n_1,l_1}]_2\,[\boldsymbol{\mathcal{E}}_{n_2,l_2}]_2\right)\,|1-(z+\mathbf{i}\,\rho)^2|^{1/2}\,d\sigma \\
= \;& \frac{1}{4}\,\frac{(n_1+1+l_1)!}{(n_1-l_1)!}\,\pi\,\delta_{n_1,n_2}\,\delta_{l_1,l_2} \\
\times \;& \left[P_{n_1}^{l_1+1}(\cosh\alpha)\,\sinh\alpha\,\cosh\alpha\,P_{n_1+1}^{l_1+1}(\cosh\alpha)-\frac{(n_1+2+l_1)}{2n_1+3}\,[P_{n_1}^{l_1+1}(\cosh\alpha)]^2\,\sinh\alpha\right] \\
+ \;& \frac{1}{4}\,(n_1+1+l_1)^2\,(n+l_1)^2\,\frac{(n_1-1+l_1)!}{(n_1+2-l_1)!}\,(1-\delta_{0,l_1})\pi\,\delta_{n_1,n_2}\,\delta_{l_1,l_2} \\
\times \;& \left[P_{n_1}^{l_1-1}(\cosh\alpha)\,\sinh\alpha\,\cosh\alpha\,P_{n_1+1}^{l_1-1}(\cosh\alpha)-\frac{(n_1+l_1)}{2n_1+3}\,[P_{n_1}^{l_1-1}(\cosh\alpha)]^2\,\sinh\alpha\right].
\end{aligned}$$

□

4 Perspectives and concluding remarks

An explicit orthogonal system of polynomial (R)-solutions over prolate spheroids has been presented. In what follows we summarize some properties of the basis polynomials, which illustrate the important role that they play in the theory of monogenic functions.

Theorem 4.1 *The monogenic polynomials* $\boldsymbol{\mathcal{E}}_{n,l}$ $(l = 0, \ldots, n+1)$ *and* $\boldsymbol{\mathcal{F}}_{n,m}$ $(m = 1, \ldots, n+1)$ *satisfy the following properties:*

1. *The polynomials* $\boldsymbol{\mathcal{E}}_{n,l}$ *and* $\boldsymbol{\mathcal{F}}_{n,m}$ *are* 2π*-periodic with respect to the variable* ϕ*;*

2. *For each* $n \in \mathbb{N}_0$*, the harmonic polynomials* $\mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l})$ $(l = 0, \ldots, n)$ *and* $\mathbf{Sc}(\boldsymbol{\mathcal{F}}_{n,m})$ $(m = 1, \ldots, n)$ *form a complete orthogonal system for the interior of the prolate spheroid (4) in the sense of the scalar product (1);*

3. *For each* $n \in \mathbb{N}_0$*, the harmonic polynomials in each of the sets*

$$\{\mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,l}),\, [\boldsymbol{\mathcal{E}}_{n,l}]_1,\, [\boldsymbol{\mathcal{E}}_{n,l}]_2 : l = 0, \ldots, n+1\}$$
$$\{\mathbf{Sc}(\boldsymbol{\mathcal{F}}_{n,m}),\, [\boldsymbol{\mathcal{F}}_{n,m}]_1,\, [\boldsymbol{\mathcal{F}}_{n,m}]_2 : m = 1, \ldots, n+1\}$$

are orthogonal for the interior of the prolate spheroid (4) in the sense of the scalar product (1);

Proof: The proof of Statement 1. involves some peculiar periodic properties of the Chebyshev polynomials of the first and second kinds, which are particularly interesting. We have then that $\boldsymbol{\mathcal{E}}_{n,l}$ and $\boldsymbol{\mathcal{F}}_{n,m}$ are periodic with period 2π with respect to the variable ϕ. Statement 2. may be found in [2], and having in mind that $\mathbf{Sc}(\boldsymbol{\mathcal{E}}_{n,n+1}) = \mathbf{Sc}(\boldsymbol{\mathcal{F}}_{n,n+1}) = 0$. The proof of Statement 3. is a consequence of Theorems 3.1.

□

Ultimately, for each degree $n \in \mathbb{N}_0$, the set $\{\boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} : l = 0, \ldots, n+1, m = 1, \ldots, n+1\}$ is formed by $2n + 3 = \dim \mathcal{R}^+(\mathcal{E}; \mathbb{R}; n)$ monogenic polynomials, and therefore, it is complete in $\mathcal{R}^+(\mathcal{E}; \mathbb{R}; n)$. Based on the orthogonal decomposition $\mathcal{R}^+(\mathcal{E}; \mathbb{R}) = \oplus \sum_{n=0}^{\infty} \mathcal{R}^+(\mathcal{E}; \mathbb{R}; n)$, and the completeness of the system in each subspace $\mathcal{R}^+(\mathcal{E}; \mathbb{R}; n)$, it follows the result.

Theorem 4.2 *For each* n*, the set of* $2n + 3$ *linearly independent monogenic polynomials*

$$\{\boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} : l = 0, \ldots, n+1, m = 1, \ldots, n+1\},$$

forms an orthogonal basis in the subspace $\mathcal{R}^+(\mathcal{E}; \mathbb{R}; n)$ *in the sense of the scalar product* (1)*. Consequently,*

$$\{\boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} : l = 0, \ldots, n+1, m = 1, \ldots, n+1; n = 0, 1, \ldots\}$$

is an orthogonal basis in $\mathcal{R}^+(\mathcal{E}; \mathbb{R})$.

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