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Second-order Numerical Scheme for Singularly Perturbed Reaction–Diffusion Robin Problems[1]

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Abstract: In this article, we consider singularly perturbed reaction-diffusion Robin boundary-value problems. To solve these problems we construct a numerical method which involves both the cubic spline and classical finite difference schemes. The proposed scheme is applied on a piece-wise uniform Shishkin mesh. Truncation error is obtained, and the stability of the method is analyzed. Also, parameter-uniform errors estimates are derived. Three test problems are experimented.

Keywords: Singular perturbation problems, Robin boundary conditions, cubic spline, piecewise-uniform Shishkin mesh, error analysis.

Mathematics Subject Classification: AMS 65L10, CR G1.7

1 Introduction

Singular perturbation problems (SPPs) occur in many fields of applied mathematics, and engineering, such as fluid flows at high Reynolds numbers, heat and mass transfer at high Péclet numbers, semiconductor devices, etc. The solution of these problems has a multi-scale character, it has two components, one varies slowly and the other varies fastly, so that the use of standard central finite difference schemes often yield numerical solutions with spurious non-physical oscillations. Hence, some special treatments are required for the numerical solution of these problems which attract researchers for the past few decades.

Let us consider the following singularly perturbed reaction–diffusion boundary–value problem (BVP):

$$Lu(x) \equiv -\varepsilon u''(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1) \tag{1}$$

$$\alpha_1 u(0) - \beta_1 u'(0) = A, \quad \alpha_2 u(1) + \beta_2 u'(1) = B, \tag{2}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$ and $\varepsilon > 0$ is a small parameter, b and f are sufficiently smooth functions, such that $b(x) \geq \beta > 0$ on $\overline{\Omega} = [0,1]$. Under these assumptions, the BVP (1-2) possesses a unique

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solution $u(x) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$. In general, the solution $u(x)$ may exhibit two boundary layers of exponential type at both end points $x = 0, 1$. BVPs of the type (1-2) model confinement of a plasma column by reaction pressure and geophysical fluid dynamics [3].

Various numerical methods are proposed for SPPs in the literature, more details can be found in the books of Farrell et al. [5], and Roos et al. [13]. In [11], Natesan and Ramanujam provided a booster method to the BVPs of the form (1) subject to Neumann boundary conditions, which incorporates an asymptotic expansion into any numerical method and give higher order accuracy. Kadalbajoo and Bawa [7, 8] used cubic spline on variable mesh for solving SPPs. Natesan et al. devised a domain decomposition method for singularly perturbed nonself-adjoint BVPs and implemented in a parallel computer in [12, 15]. An efficient numerical method for SPPs is presented in [16]. Recently, the authors developed a computational method using quintic spline for reaction–diffusion Dirichlet BVPs in [1]. An High-Order Difference Approximation with Identity Expansions (HODIE) scheme for singularly perturbed sefl-adjoint BVPs is obtained in [6]. A parameter-uniform numerical scheme has been proposed and analyzed in [10] for self-adjoint SPP with Dirichlet boundary conditions.

In this article, we propose a hybrid numerical scheme for the BVP (1-2). First of all we derive a difference scheme using cubic spline on variable meshes. In the literature of numerical methods for SPPs, piece-wise uniform Shishkin mesh plays a prominent role [9, 5], and here we use the same mesh for our cubic spline scheme. The basic idea behind the Shishkin mesh is to divide the domain $\overline{\Omega} = [0, 1] = [0, \sigma) \cup [\sigma, 1 - \sigma] \cup (1 - \sigma, 1]$ and place $N/4$ mesh points in the boundary layer regions $[0, \sigma)$ and $(1 - \sigma, 1]$ and $N/2$ intervals in the regular region $[\sigma, 1 - \sigma]$, where the transition parameter σ will be a function of N and ε. The newly developed cubic spline scheme satisfies the discrete maximum principle only in the boundary layer regions and not in the regular region. As an outcome the cubic spline solution is not stable in the outer region. This is because the mesh intervals are coarse in the regular region and to satisfy the discrete maximum principle one has to restrict the mesh size in relation with the parameter ε. To overcome this difficulty, we use the cubic spline scheme only in the boundary layer regions and the classical finite difference scheme in the regular regions. Truncation errors are derived, and the present method provides second–order uniform convergent result throughout the domain of interest. Three numerical examples have been carried out to show the efficiency of the method.

The paper is organized in the following style: The cubic spline difference scheme and the hybrid scheme are derived in Section 2, and the piecewise uniform meshes are also presented. Convergence results are obtained in Section 3. Semilinear problems are treated in Section 4. Section 5 presents numerical examples. The paper ends with conclusions. Throughout this paper C denotes a generic positive constant independent of ε, x_i and N, although its value differ from place to place.

2 Numerical Scheme

In this section, first, we derive the cubic spline scheme on variable meshes, and then devise the hybrid scheme on the piece-wise uniform Shishkin meshes for the SPP (1-2).

2.1 Difference scheme based on cubic spline

Let the mesh points of $\overline{\Omega} = [0, 1]$ be

$$x_0 = 0,\ x_i = \sum_{k=0}^{i-1} h_k,\ h_k = x_{k+1} - x_k,\ x_N = 1,\ i = 1, 2, \ldots, N-1. \tag{3}$$

We derive the difference scheme in the following.

For given values $u(x_0), u(x_1), \cdots, u(x_N)$ of a function $u(x)$ at the nodal points $x_0, x_1, \ldots, x_N$, there exists an interpolating cubic spline $S(x)$ with the following properties:

(i) $S(x)$ coincides with a polynomial of degree three on each subinterval $[x_i, x_{i+1}], i = 0, \ldots, N-1$;

(ii) $S(x) \in \mathcal{C}^2[0,1]$;

(iii) $S(x_i) = u(x_i),\ i = 0, \ldots, N$.

The cubic spline can be given by

$$\begin{aligned} S(x) = \quad & \frac{(x_{i+1}-x)^3}{6h_i} M_i + \frac{(x-x_i)^3}{6h_i} M_{i+1} + \left(u(x_i) - \frac{h_i^2}{6} M_i\right)\left(\frac{x_{i+1}-x}{h_i}\right) + \\ & + \left(u(x_{i+1}) - \frac{h_i^2}{6} M_{i+1}\right)\left(\frac{x-x_i}{h_i}\right), \quad x_i \le x \le x_{i+1},\ i = 0, \cdots, N-1 \end{aligned} \tag{4}$$

where $M_i = S''(x_i),\ i = 0, \cdots, N$.

The first derivative of $S(x)$ is given by

$$\begin{aligned} S'(x) = -M_i \frac{(x_{i+1}-x)^2}{2h_i} + M_{i+1} \frac{(x-x_i)^2}{2h_i} + \frac{u(x_{i+1}) - u(x_i)}{h_i} - \frac{(M_{i+1} - M_i)}{6} h_i, \\ x_i \le x \le x_{i+1},\ i = 0, \cdots, N-1. \end{aligned} \tag{5}$$

and the second derivative is

$$S''(x) = M_i \frac{(x_{i+1}-x)}{h_i} + M_{i+1} \frac{(x-x_i)}{h_i}. \tag{6}$$

For the one sided limit of the first derivative, from (5), we have

$$S'(x_i-) = \frac{h_{i-1}}{6} M_{i-1} + \frac{h_{i-1}}{3} M_i + \frac{u(x_i) - u(x_{i-1})}{h_{i-1}}, \tag{7}$$

and

$$S'(x_i+) = -\frac{h_i}{3} M_i - \frac{h_i}{6} M_{i+1} + \frac{u(x_{i+1}) - u(x_i)}{h_i}. \tag{8}$$

From (4) and (6), the functions $S(x)$ and $S''(x)$ are continuous on $\overline{\Omega}$ and for $S'(x)$ to be continuous at the interior nodes x_i, we have from (7)-(8), the following well-known 'continuity condition':

$$\frac{h_{i-1}}{6} M_{i-1} + \left(\frac{h_i + h_{i-1}}{3}\right) M_i + \frac{h_i}{6} M_{i+1} = \left(\frac{u(x_{i+1}) - u(x_i)}{h_i}\right) - \left(\frac{u(x_i) - u(x_{i-1})}{h_{i-1}}\right),\ i = 1, \cdots, N-1. \tag{9}$$

Equation (9) ensures the continuity of the first order derivative of the spline $S(x)$ at the interior nodes.

Substituting

$$-\varepsilon M_j + b(x_j) u(x_j) = f(x_j), \quad j = i, i \pm 1, \tag{10}$$

in (9), we obtain the following system:

$$\begin{cases} \left[\dfrac{-3\varepsilon}{h_{i-1}(h_i + h_{i-1})} + \dfrac{h_{i-1}}{2(h_i + h_{i-1})} b_{i-1}\right] u_{i-1} + \left[\dfrac{3\varepsilon}{h_i h_{i-1}} + b_i\right] u_i + \\ + \left[\dfrac{-3\varepsilon}{h_i(h_i + h_{i-1})} + \dfrac{h_i}{2(h_i + h_{i-1})} b_{i+1}\right] u_{i+1} = \left[\dfrac{h_{i-1}}{2(h_i + h_{i-1})}\right] f_{i-1} + f_i + \left[\dfrac{h_i}{2(h_i + h_{i-1})}\right] f_{i+1}. \end{cases} \tag{11}$$

Now, using expressions (7) and (8) for approximation of the first derivative at boundary points, we obtain by following:

$$\begin{cases} \left[\dfrac{3\varepsilon}{h_0}\left(\alpha_1+\dfrac{\beta_1}{h_0}\right)+b_0\beta_1\right]u_0+\left[-\dfrac{3\varepsilon\beta_1}{h_0^2}+\dfrac{b_1}{2}\beta_1\right]u_1=\dfrac{3\varepsilon A}{h_0}+\beta_1 f_0+\dfrac{\beta_1}{2}f_1; \\ \left[-\dfrac{3\varepsilon\beta_2}{h_{N-1}^2}+\dfrac{b_{N-1}}{2}\beta_2\right]u_{N-1}+\left[\dfrac{3\varepsilon}{h_{N-1}}\left(\alpha_2+\dfrac{\beta_2}{h_{N-1}}\right)+b_N\beta_2\right]u_N=\dfrac{3\varepsilon B}{h_{N-1}}+\dfrac{\beta_2}{2}f_{N-1}+\beta_2 f_N. \end{cases} \tag{12}$$

Finally, the equations (11) and (12) constitute the system of linear algebraic equations, which gives the approximations $u_0, u_1, \cdots, u_N$ of the solution $u(x)$ at $x_0, x_1, \cdots, x_N$.

2.2 Piece-wise uniform Shishkin mesh

The cubic spline difference scheme derived in the previous subsection 2.1 is on variable meshes and it is a more general one. For SPPs one need finer mesh in the boundary layer regions and coarse mesh in the regular region which can be easily obtained *viz.* the piece-wise uniform Shishkin mesh. More precisely, the domain $\overline{\Omega}$ is divided into three subintervals as

$$\overline{\Omega}=[0,\sigma]\cup[\sigma,1-\sigma]\cup[1-\sigma,1],$$

for some σ such that $0<\sigma\leq 1/4$. On the subintervals $[0,\sigma),(1-\sigma,1]$ a uniform mesh with $N/4$ mesh–intervals is placed, where $[\sigma,1-\sigma]$ has a uniform mesh with $N/2$ mesh intervals. It is obvious that the mesh is uniform when $\sigma=1/4$, and it is fitted to the problem by choosing σ be the following function of N, ε and σ_0

$$\sigma=\min\left\{\frac{1}{4},\sigma_0\sqrt{\varepsilon}\ln N\right\}, \tag{13}$$

where $\sigma_0\geq 2/\sqrt{\beta}>0$ is a constant. Further, we denote the mesh size in the regions $[0,\sigma),(1-\sigma,1]$ as $h^{(1)}=4\sigma/N$, and in $[\sigma,1-\sigma]$ by $h^{(2)}=2(1-2\sigma)/N$.

2.3 The hybrid scheme

The cubic spline difference scheme (11-12) is applied on the Shishkin mesh and we found that the resulting stiffness matrix is not a M-matrix. The loss of M-matrix property is mainly from the coarse mesh used in the outer region $[\sigma,1-\sigma]$. And as a consequence the corresponding difference operator fails to satisfy the discrete maximum principle only in the outer region. In order make the discrete operator to satisfy the discrete maximum principle, one has to use smaller meshes in the outer region. To over come this difficulty, we replace the unstable cubic spline scheme by some other stable scheme in the outer region. Hence, we keep the cubic spline scheme for the boundary layer regions $[0,\sigma),(1-\sigma,1]$, and replace the cubic spline scheme by classical central finite difference scheme in the outer region. More precisely, the new hybrid scheme is given by

$$L^N u_i\equiv r_i^- u_{i-1}+r_i^c u_i+r_i^+ u_{i+1}=q_i^- f_{i-1}+q_i^c f_i+q_i^+ f_{i+1},\quad i=1,\cdots,N-1, \tag{14}$$

along with the following equations corresponding to the boundary points

$$\begin{cases} r_0^c u_0+r_0^+ u_1=q_0^- +q_0^c f_0+q_0^+ f_1, \\ r_N^- u_{N-1}+r_N^c u_0=q_N^- +q_N^c f_{N-1}+q_N^+ f_N, \end{cases} \tag{15}$$

for $i = 1, \cdots, N/4-1$ and $3N/4+1, \cdots, N-1$

$$\begin{cases} r_i^- = \dfrac{-3\varepsilon}{h_{i-1}(h_i+h_{i-1})} + \dfrac{h_{i-1}}{2(h_i+h_{i-1})}b_{i-1}; \quad r_i^c = \dfrac{3\varepsilon}{h_i h_{i-1}} + b_i; \\ r_i^+ = \dfrac{-3\varepsilon}{h_i(h_i+h_{i-1})} + \dfrac{h_i}{2(h_i+h_{i-1})}b_{i+1}; \\ q_i^- = \dfrac{h_{i-1}}{2(h_i+h_{i-1})}; \quad q_i^c = 1; \quad q_i^+ = \dfrac{h_i}{2(h_i+h_{i-1})}, \end{cases} \tag{16}$$

and for $i = N/4, \cdots, 3N/4$

$$\begin{cases} r_i^- = \dfrac{-2\varepsilon}{h_{i-1}(h_i+h_{i-1})}; \quad r_i^c = \dfrac{2\varepsilon}{h_i h_{i-1}} + b_i; \quad r_i^+ = \dfrac{-2\varepsilon}{h_i(h_i+h_{i-1})}; \\ q_i^- = 0; \quad q_i^c = 1; \quad q_i^+ = 0. \end{cases} \tag{17}$$

and

$$\begin{cases} r_0^c = \dfrac{3\varepsilon}{h_0}\left(\alpha_1 + \dfrac{\beta_1}{h_0}\right) + b_0\beta_1; \quad r_0^+ = -\dfrac{3\varepsilon\beta_1}{h_0^2} + \dfrac{b_1}{2}\beta_1; \\ q_0^- = \dfrac{3\varepsilon A}{h_0}; \quad q_0^c = \beta_1; \quad q_0^+ = \dfrac{\beta_1}{2}; \\ r_N^- = -\dfrac{3\varepsilon\beta_2}{h_{N-1}^2} + \dfrac{b_{N-1}}{2}\beta_2; \quad r_N^c = \dfrac{3\varepsilon}{h_{N-1}}\left(\alpha_2 + \dfrac{\beta_2}{h_{N-1}}\right) + b_N\beta_2; \\ q_N^- = \dfrac{3\varepsilon B}{h_{N-1}}; \quad q_N^c = \dfrac{\beta_2}{2}; \quad q_N^+ = \beta_2. \end{cases} \tag{18}$$

The tri-diagonal system of linear algebraic equations (14-15) can be solved by any existing codes.

3 Convergence Analysis

In this section, first we derive the truncation at the nodal points depending on its location in the domain, *i.e.,* in the boundary layer region, outer region, the transition points, and the boundary points. And then we show the stability of the proposed method. Finally, we derive the ε–uniform second-order error estimate.

For $i = 1, \cdots, N/4-1$ and $3N/4+1, \cdots, N-1$, the truncation error of the hybrid scheme is given by

$$\tau_{i,u} = [r_i^- u(x_{i-1}) + r_i^c u(x_i) + r_i^+ u(x_{i+1})] - [q_i^- f(x_{i-1}) + q_i^c f(x_i) + q_i^+ f(x_{i+1})]. \tag{19}$$

Using the differential equation (1) for f in the above expression, we get

$$\begin{aligned} \tau_{i,u} = [r_i^- u(x_{i-1}) + r_i^c u(x_i) + r_i^+ u(x_{i+1})] - [q_i^-(-\varepsilon u''(x_{i-1}) + b_{i-1}u(x_{i-1})) + \\ + q_i^c(-\varepsilon u''(x_i) + b_i u(x_i)) + q_i^+(-\varepsilon u''(x_{i+1}) + b_{i+1}u(x_{i+1}))]. \end{aligned} \tag{20}$$

Now, making use of the Taylor series expansion, we have

$$u(x_{i-1}) = u(x_i) - h_{i-1}u'(x_i) + \frac{h_{i-1}^2}{2!}u''(x_i) - \frac{h_{i-1}^3}{3!}u^{(iii)}(x_i) + \frac{h_{i-1}^4}{4!}u^{(iv)}(x_i) + \cdots,$$

and

$$u(x_{i+1}) = u(x_i) + h_i u'(x_i) + \frac{h_i^2}{2!}u''(x_i) + \frac{h_i^3}{3!}u^{(iii)}(x_i) + \frac{h_i^4}{4!}u^{(iv)}(x_i) + \cdots .$$

Using the values of $u(x_{i-1})$, $u(x_{i+1})$ in (20), we have

$$\tau_{i,u} = T_{0,i}u(x_i) + T_{1,i}u'(x_i) + T_{2,i}u''(x_i) + T_{3,i}u^{(iii)}(x_i) + T_{4,i}u^{(iv)}(x_i) + \text{h.o.t.}, \tag{21}$$

where

$$\begin{aligned}
T_{0,i} &= r_i^- + r_i^c + r_i^+ - (q_i^- b_{i-1} + q_i^c b_i + q_i^+ b_{i+1}),\\
T_{1,i} &= -h_{i-1}r_i^- + h_i r_i^+ + (h_{i-1}q_i^- b_{i-1} - h_i q_i^+ b_{i+1}),\\
T_{2,i} &= \frac{h_{i-1}^2}{2!}r_i^- + \frac{h_i^2}{2!}r_i^+ + \varepsilon(q_i^- + q_i^c + q_i^+) - \left(\frac{h_{i-1}^2}{2!}q_i^- b_{i-1} + \frac{h_i^2}{2!}q_i^+ b_{i+1}\right),\\
T_{3,i} &= -\frac{h_{i-1}^3}{3!}r_i^- + \frac{h_i^3}{3!}r_i^+ - \varepsilon(q_i^- h_{i-1} - q_i^+ h_i) + \left(\frac{h_{i-1}^3}{3!}q_i^- b_{i-1} - \frac{h_i^3}{3!}q_i^+ b_{i+1}\right),\\
T_{4,i} &= \frac{h_{i-1}^4}{4!}r_i^- + \frac{h_i^4}{4!}r_i^+ + \varepsilon(q_i^- \frac{h_{i-1}^2}{2!} + \frac{h_i^2}{2!}q_i^+) - \left(\frac{h_{i-1}^4}{4!}q_i^- b_{i-1} + \frac{h_i^4}{4!}q_i^+ b_{i+1}\right).
\end{aligned}$$

It can be easily seen that

$$T_{0,i} = T_{1,i} = T_{2,i} = T_{3,i} = 0, \quad T_{4,i} = -3\varepsilon\left(\frac{h_i^3 + h_{i-1}^3}{h_i + h_{i-1}}\right)\left[\frac{1}{4!} - \frac{1}{2!6}\right].$$

Thus, we have

$$\tau_{i,u} = -3\varepsilon\left(\frac{h_i^3 + h_{i-1}^3}{h_i + h_{i-1}}\right)\left[\frac{1}{4!} - \frac{1}{2!6}\right]u^{(iv)}(x_i) + O(N^{-3}). \tag{22}$$

For $i = N/4, \cdots, 3N/4$, we can proceed in a similar manner to show that

$$\tau_{i,u} = -\varepsilon\left(\frac{h_i - h_{i-1}}{3}\right)u^{(iii)}(x_i) + \frac{2\varepsilon}{4!}\left(\frac{h_i^3 + h_{i-1}^3}{h_i + h_{i-1}}\right)u^{(iv)}(x_i) + O(N^{-3}). \tag{23}$$

Note that the above expression for truncation error in outer region for points other than transition points (i.e, for $i \neq N/4, 3N/4$) can also be represented as

$$\tau_{i,u} = \frac{2\varepsilon}{h_{i-1} + h_i}\left(\frac{R_3(x_i, x_{i-1}, u)}{h_{i-1}} + \frac{R_3(x_i, x_{i+1}, u)}{h_i}\right), \tag{24}$$

where $R_n(a,p,g) = \dfrac{1}{n!}\displaystyle\int_a^p (p-\xi)g^{(n+1)}(\xi)d\xi$ denotes the remainder obtained from the Taylor expansion in integral form.

For $i = N/4$, $3N/4$, the truncation error can be given as

$$\tau_{i,u} = \frac{2\varepsilon}{h_{i-1} + h_i}\left(\frac{R_2(x_i, x_{i-1}, u)}{h_{i-1}} + \frac{R_2(x_i, x_{i+1}, u)}{h_i}\right). \tag{25}$$

The truncation error at the boundary point x_0 is given by

$$\tau_{0,u} = r_0^c u(x_0) + r_0^+ u(x_1) - q_0^- - q_0^c f_0 - q_0^+ f_1 \tag{26}$$

Again using (1), and the Taylor series expansion for $u(x_1)$, we have

$$\tau_{0,u} = -q_0^c + T_{0,0}u(x_0) + T_{1,0}u'(x_0) + T_{2,0}u''(x_0) + T_{3,0}u^{(iii)}(x_0) + T_{4,0}u^{(iv)}(x_0) + \text{h.o.t.}, \quad (27)$$

where

$$\begin{aligned} T_{0,0} &= r_0^c + r_i^+ - q_0^c b_0 + q_0^+ b_1, \\ T_{1,0} &= h_0 r_0^+ - h_0 q_0^+ b_1, \\ T_{2,0} &= \frac{h_0^2}{2!} r_0^+ + \varepsilon(q_0^c + q_i^+) - \frac{h_0^2}{2!} q_0^+ b_1, \\ T_{3,0} &= \frac{h_0^3}{3!} r_0^+ + \varepsilon h_0 q_0^c - \frac{h_0^3}{3!} q_0^+ b_1, \\ T_{4,0} &= \frac{h_0^4}{4!} r_0^+ + \varepsilon \frac{h_0^2}{2!} q_0^+ - \frac{h_0^4}{4!} q_0^+ b_1. \end{aligned}$$

It can be seen easily that

$$-q_0^c + T_{0,0}u(x_0) + T_{1,0}u'(x_0) = T_{2,0} = T_{3,0} = 0, \quad T_{4,0} = -\varepsilon\beta_1 h_0^2 \left(\frac{1}{4!} - \frac{1}{2!6}\right).$$

Therefore, the truncation error at x_0 can be given by

$$\tau_{0,u} = -\varepsilon\beta_1 h_0^2 \left(\frac{1}{4!} - \frac{1}{2!6}\right) u_0^{(iv)}(x_0) + O(N^{-3}). \quad (28)$$

Similarly the truncation error at x_N can be obtained as

$$\tau_{N,u} = -\varepsilon\beta_2 h_{N-1}^2 \left(\frac{1}{4!} - \frac{1}{2!6}\right) u_N^{(iv)}(x_N) + O(N^{-3}). \quad (29)$$

Proposition 3.1 *Let $u(x)$ and u_i be respectively the solutions of (1-2) and (14-15). Then, the local truncation error satisfies the following bounds:*

$$\begin{aligned} |\tau_{i,u}| &\le CN^{-2}\sigma_0^2 \ln^2 N, \quad \textit{for} \quad 0 \le i < N/4 \quad \textit{and} \quad 3N/4 < i \le N, \\ |\tau_{i,u}| &\le C(N^{-2}\varepsilon + N^{-\sqrt{\beta}\sigma_0}), \quad \textit{for} \quad N/4 < i < 3N/4, \\ |\tau_{i,u}| &\le C(N^{-3} + N^{-\sqrt{\beta}\sigma_0}), \quad \textit{for} \quad i = N/4, 3N/4 \quad \textit{and} \quad h^{(2)} \ge \sqrt{\varepsilon}, \\ |\tau_{i,u}| &\le C(N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0}), \quad \textit{for} \quad i = N/4, 3N/4 \quad \textit{and} \quad h^{(2)} < \sqrt{\varepsilon}. \end{aligned}$$

Proof. We distinguish several cases depending on the location of the mesh points. Before starting the proof, let us state the bounds for the derivatives of the continuous solution; the solution $u(x)$ of the BVP (1-2) satisfies the following bound

$$|u^{(k)}(x)| \le C\left[1 + \varepsilon^{(-k/2)} e(x, x, \beta, \varepsilon)\right], \quad (30)$$

where $e(x, x, \beta, \varepsilon) = \exp(-x\sqrt{\beta/\varepsilon}) + \exp(-(1-x)\sqrt{\beta/\varepsilon})$.

For $x_i \in [0, \sigma) \cup (1-\sigma, 1]$ from (22), we get

$$|\tau_{i,u}| \le C\left[\left(h^{(1)}\right)^2 \varepsilon + \left(h^{(1)}\right)^2 \varepsilon^{-1}[e(x_i, x_{i+1}, \beta, \varepsilon) + e(x_{i-1}, x_i, \beta, \varepsilon)]\right]. \quad (31)$$

Using $h^{(1)} = 4N^{-1}\sigma_0\sqrt{\varepsilon}\ln N$ and bounding the exponential functions by constants. It is easy to show that $|\tau_{i,u}| \le CN^{-2}\sigma_0^2 \ln^2 N$, for $0 \le i < N/4$ and $3N/4 < i \le N$.

For $x_i \in [\sigma, 1-\sigma]$, we discuss two cases: First, if $\left(h^{(2)}\right)^2 < \varepsilon$, using (23) and the bounds of the derivatives of $u(x)$ from (30) , we obtain

$$\begin{aligned} |\tau_{i,u}| &\leq C\left[\left(h^{(2)}\right)^2\varepsilon + \left(h^{(2)}\right)^2\varepsilon^{-1}(e(x_i, x_{i+1}, \beta, \varepsilon) + e(x_{i-1}, x_i, \beta, \varepsilon))\right] \\ &\leq C(N^{-2}\varepsilon + N^{-\sqrt{\beta}\sigma_0}). \end{aligned} \tag{32}$$

Secondly, if $\left(h^{(2)}\right)^2 \geq \varepsilon$, again, using (24) and the bounds of the derivatives of $u(x)$ from (30), one can obtain the following

$$|\tau_{i,u}| \leq C\left(\left(h^{(2)}\right)^2\varepsilon + \int_{x_i}^{x_{i+1}}(x_{i+1}-\xi)\varepsilon^{-1}e(\xi,\xi,\beta,\varepsilon)d\xi + \int_{x_{i-1}}^{x_i}(\xi - x_{i-1})\varepsilon^{-1}e(\xi,\xi,\beta,\varepsilon)d\xi\right). \tag{33}$$

Integrating the first integral by parts, we get

$$\begin{aligned} \int_{x_i}^{x_{i+1}}(x_{i+1}-\xi)\varepsilon^{-1}e(\xi,\xi,\beta,\varepsilon)d\xi &\leq C\left(h^{(2)}\varepsilon^{-1/2}e(x_i,x_i,\beta,\varepsilon) + \int_{x_i}^{x_{i+1}}\varepsilon^{-1/2}e(\xi,\xi,\beta,\varepsilon)d\xi\right) \\ &\leq C\left[e(x_i,x_i,\beta,\varepsilon) + e(x_i,x_{i+1},\beta,\varepsilon)\right] \\ &\leq CN^{-\sqrt{\beta}\sigma_0}. \end{aligned}$$

A bound for the second integral can be found in similar fashion. Using that $h^{(2)} < 2N^{-1}$, we get

$$|\tau_{i,u}| \leq C\left(N^{-2}\varepsilon + N^{-\sqrt{\beta}\sigma_0}\right). \tag{34}$$

Finally, we analyze the error for the transition point $x_{N/4} = \sigma$ (similarly, for $x_{3N/4} = 1-\sigma$).

If $h^{(2)} \geq \sqrt{\varepsilon}$, then using (25), we get

$$\begin{aligned} |\tau_{N/4,u}| &\leq C\left[h^{(2)}\varepsilon + \int_{x_{N/4}}^{x_{N/4+1}}\varepsilon^{-1/2}e(\xi,\xi,\beta,\varepsilon)d\xi + \int_{x_{N/4-1}}^{x_{N/4}}\varepsilon^{-1/2}e(\xi,\xi,\beta,\varepsilon)d\xi\right] \\ &\leq C(N^{-3} + e(x_{N/4}, x_{N/4+1}, \beta, \varepsilon) + e(x_{N/4-1}, x_{N/4}, \beta, \varepsilon)) \\ &\leq C(N^{-3} + N^{-\sqrt{\beta}\sigma_0}). \end{aligned}$$

On the other hand, If $h^{(2)} < \sqrt{\varepsilon}$,

$$\begin{aligned} |\tau_{N/4,u}| &\leq C\left(h^{(2)}\varepsilon + h^{(2)}\varepsilon^{-1/2}[e(x_{N/4}, x_{N/4+1}, \beta, \varepsilon) + e(x_{N/4-1}, x_{N/4}, \beta, \varepsilon)]\right) \\ &\leq N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0}. \end{aligned} \tag{35}$$

Combining all the previous results, we obtain the required truncation error. ■

Definition 3.2 (Discrete Maximum principle) *If the stiffness matrix associated with the hybrid scheme (14) is an M-matrix, then it satisfies that* $L^N[v_h] \geq 0 \Rightarrow v_h \geq 0$, *for all discrete functions* v_h *defined on* $\overline{\Omega}$.

Proposition 3.3 given below shows that the stiffness matrix of the present method is an M-matrix, and hence, it satisfies the discrete maximum principle, and as a consequence, the discrete stability estimate.

Proposition 3.3 *For sufficiently large N, and $16N^{-2}\sigma_0^2 \ln^2 N\beta^* < 6$, where $\beta^* = \max_{0\le i\le N} b(x_i)$. Then, for $i = 1, 2, \cdots, N$, we have*

$$r_i^- < 0, \quad r_i^+ < 0, \quad |r_i^c| - |r_i^-| - |r_i^+| \ \ge\ \min(1, \beta) > 0.$$

Also, we have

$$r_0^+ < 0, \quad r_N^- < 0, \quad |r_0^c| - |r_0^+| \ \ge\ 0, |r_N^c| - |r_N^-| \ \ge\ 0.$$

Proof. Clearly, $r_i^- < 0$ and $r_i^+ < 0$, for $i = N/4, \cdots, 3N/4$.

For $r_i^- < 0$, $i = 1, \cdots, N/4 - 1$ and $i = 3N/4 + 1, \cdots, N - 1$, we have

$$r_i^- \;=\; \frac{-3\varepsilon}{h_{i-1}(h_i + h_{i-1})} + \frac{h_{i-1}}{2(h_i + h_{i-1})} b_{i-1} \;=\; \frac{-3\varepsilon}{2\left(h^{(1)}\right)^2} + \frac{1}{4} b_{i-1}$$

since $h^{(1)} = 4N^{-1}\sigma_0\sqrt{\varepsilon}\ln N$ and $b_{i-1} \le \beta^*$, we have $r_i^- < 0$.
For the left boundary point

$$r_0^+ \;=\; \frac{-3\varepsilon\beta_1}{h_0^2} + \frac{b_1}{2}\beta_1 \;=\; \frac{-3\varepsilon\beta_1}{(h^{1)})^2} + \frac{b_1}{2}\beta_1.$$

Again, since $h^{(1)} = 4N^{-1}\sigma_0\sqrt{\varepsilon}\ln N$ and $b_1 \le \beta^*$, we have $r_0^+ < 0$.
Corresponding to the right boundary point

$$r_N^- = \frac{-3\varepsilon\beta_2}{h_{N-1}^2} + \frac{b_{N-1}}{2}\beta_2 \;=\; \frac{-3\varepsilon\beta_2}{(h^{(1)})^2} + \frac{b_{N-1}}{2}\beta_2 \tag{36}$$

Using $h^{(1)} = 4N^{-1}\sigma_0\sqrt{\varepsilon}\ln N$ and $b_{N-1} \le \beta^*$, we have $r_N^- < 0$.

Similarly, it can be shown that $r_i^+ < 0$ for $i = 1, \cdots, N/4 - 1$ and $i = 3N/4 + 1, \cdots, N - 1$.
Now, clearly for $i = N/4, \cdots, 3N/4$

$$|r_i^c| - |r_i^-| - |r_i^+| \ \ge\ \beta.$$

For $i = 1, \cdots, N/4 - 1$ and $i = 3N/4 + 1, \cdots, N$, we have

$$\begin{aligned}
|r_i^c| - |r_i^-| - |r_i^+| \;&=\; \frac{3\varepsilon}{h_i h_{i-1}} + b_i - \frac{3\varepsilon}{h_{i-1}(h_i + h_{i-1})} + \frac{h_{i-1}}{2(h_i + h_{i-1})} b_{i-1} - \\
&\quad - \frac{3\varepsilon}{h_i(h_i + h_{i-1})} + \frac{h_i}{2(h_i + h_{i-1})} b_{i+1} \\
&=\; b_i + \frac{h_{i-1}}{2(h_i + h_{i-1})} b_{i-1} + \frac{h_i}{2(h_i + h_{i-1})} b_{i+1} \\
&\ge\; \beta + \left[\frac{h_{i-1}}{2(h_i + h_{i-1})} + \frac{h_i}{2(h_i + h_{i-1})}\right]\beta = \frac{3}{2}\beta > 0.
\end{aligned}$$

Also,

$$|r_0^c| - |r_0^+| = \frac{3\varepsilon}{h_0}\left(\alpha_1 + \frac{\beta_1}{h_0}\right) + b_0\beta_1 - \frac{3\varepsilon\beta_1}{h_0^2} + \frac{b_1}{2}\beta_1$$

$$= \frac{3\varepsilon}{h_0}\alpha_1 + \left(b_0 + \frac{b_1}{2}\right)\beta_1 \ \ge\ \frac{3\varepsilon}{h_0}\alpha_1 + \frac{3}{2}\beta\beta_1 \ >\ 0.$$

and

$$|r_N^c| - |r_N^-| = \frac{3\varepsilon}{h_{N-1}}\left(\alpha_2 + \frac{\beta_2}{h_{N-1}}\right) + b_N\beta_2 - \frac{3\varepsilon\beta_2}{h_{N-1}^2} + \frac{b_{N-1}}{2}\beta_2$$

$$= \frac{3\varepsilon}{h_{N-1}}\alpha_2 + \left(\frac{b_{N-1}}{2} + b_N\right)\beta_2 \geq \frac{3\varepsilon}{h_{N-1}}\alpha_2 + \frac{3}{2}\beta\beta_2 > 0.$$

Hence, we obtain the required result. ■

Theorem 3.4 *Let $u(x)$ be the continuous solution of the BVP (1-2) and u_i be the numerical solution of (1-2) obtained from the difference scheme as given in (14-15). Then,*

$$|u(x_i) - u_i| \leq C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0}\right], \quad \forall x_i \in \overline{\Omega}.$$

Proof. Defining the discrete barrier function

$$\phi_i = C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0} + \frac{\sigma^2}{\sqrt{\varepsilon}}N^{-2}\psi(x_i)\right],$$

where

$$\psi(z) = \begin{cases} z/\sigma, & 0 \leq z \leq \sigma, \\ 1, & \sigma \leq z \leq 1-\sigma, \\ (1-z)/\sigma, & 1-\sigma \leq z \leq 1. \end{cases}$$

Choosing C sufficiently large, using discrete maximum principle, it is easy to see that,

$$L^N(\phi_i \pm (u(x_i) - u_i)) \geq 0,$$

equivalently,

$$L^N(\phi_i) \geq |\tau_{i,u}|$$

therefore, it follows that

$$|u(x_i) - u_i| \leq \phi_i, \quad \forall x_i \in \overline{\Omega}.$$

Thus, we have the required ε-uniform error bound. ■

Remark 3.5 *In Theorem 3.4, we obtain the error bound of order $O(N^{-1}\varepsilon)$ only at the transition points, and also only for the case $h^{(2)} < \sqrt{\varepsilon}$, which is not the practical case. Therefore, we conclude that the order of convergence is almost two (up to a logarithmic factor). Our numerical results given in Section 5 reveal the same behavior.*

4 Semilinear Problems

Consider the singularly perturbed semilinear reaction–diffusion Robin BVP:

$$-\varepsilon u_{xx} + F(x,u) = 0, \quad x \in \Omega = (0,1) \tag{37}$$

$$\alpha_1 u(0) - \beta_1 u'(0) = A, \quad \alpha_2 u(1) + \beta_2 u'(1) = B, \tag{38}$$

where $F(x,u)$ is a smooth function such that

$$F_u(x,u) \geq \beta > 0, \quad (x,u) \in \overline{\Omega} \times \mathbb{R}. \tag{39}$$

Assume that the reduced problem $F(x, u_0) = 0$ has a unique solution $u_0 \in \mathcal{C}^2(\overline{\Omega})$. Then, the BVP (37-38) has a unique solution. Here, the notation u_{xx} is used instead of u'' for the sake of convenience. Chang and Howes [2] studied the theoretical aspects of the semilinear BVP (37-38).

To obtain the numerical solution of (37-38), the Newton method of quasilinearization is applied to obtain a sequence $\{u^m\}_0^\infty$ of approximations with a proper choice of the initial guess $u^0(x)$. Then, define u^{m+1}, for each fixed nonnegative integer m, to be the solution of the following linear problem:

$$-\varepsilon u_{xx}^{m+1} + b^m(x)u^{m+1} = F^m(x), \quad x \in \Omega \tag{40}$$

$$\alpha_1 u(0) - \beta_1 u'(0) = A, \quad \alpha_2 u(1) + \beta_2 u'(1) = B, \tag{41}$$

where

$$b^m(x) = F_u(x, u^m), \quad F^m(x) = F_u(x, u^m)u^m - F(x, u^m).$$

From (39), it follows that, for each fixed m,

$$b^m(x) = F_u(x, u^m) \geq \beta > 0, \quad (x, u^m) \in \overline{\Omega} \times \mathbb{R}.$$

If the initial guess $u^0(x)$ is sufficiently close to the solution $u(x)$ then following the method of proof given in [4] one can prove that the sequence $\{u^m\}_0^\infty$ converges to $u(x)$.

For each fixed m, the BVP (40-41) is a linear BVP of the form (1-2), and hence can be solved by the hybrid scheme given in (14-15). For the Newton quasilinearization process, the following convergence criteria is used:

$$|u^{m+1}(x_j) - u^m(x_j)| \leq \text{TOL}, \quad x_j \in \overline{\Omega}, \quad m \geq 0, \tag{42}$$

where $u^m(x_j)$ is the mth iteration solution at the jth mesh point and TOL is the prescribed tolerance bound. From the Example 5.3 given in the next section, it has been seen that for a fixed TOL, the number of iterations required to satisfy (42) is independent of ε (refer Table 3 - see the Appendix).

5 Numerical Experiments

To show the accuracy of the present method, it has been applied to three test problems; two linear problems and one semilinear problem. To calculate the maximum point–wise error and rate of convergence, we use the double mesh principle. For this, we shall compute U^N and U^{2N} which are described as follows:

Let $\overline{\Omega}_\varepsilon^N$ be a Shishkin mesh with the parameter σ as defined in (13) altered slightly to

$$\widetilde{\sigma} = \min\left\{\frac{1}{4}, \sigma_0\sqrt{\varepsilon}\ln\left(\frac{N}{2}\right)\right\}.$$

Then, for $i = 0, 1, \cdots, N$, the ith point of the mesh $\overline{\Omega}_\varepsilon^N$ coincides with the $(2i)$th point of the mesh $\overline{\Omega}_\varepsilon^{2N}$. This idea is mainly to obtain better comparison between the solutions from the N, $2N$ mesh intervals. More precisely, in the case of $2N$ mesh intervals, if we use the transition parameter σ as given in (13), then the value of σ will be different in the cases of N and $2N$ mesh intervals, and the comparison of the solutions will not be accurate. Therefore, to fix σ, we made this alternate new definition.

Define the double mesh differences to be

$$E_\varepsilon^N = \max_{x_i \in \overline{\Omega}_\varepsilon^N} |U^N(x_j) - U^{2N}(x_j)|, \quad \text{and} \quad E^N = \max_\varepsilon E_\varepsilon^N,$$

where $U^N(x_j)$ and $U^{2N}(x_j)$ respectively denote the numerical solutions obtained using N and $2N$ mesh intervals. Further, we calculate the parameter-robust orders of convergence as

$$p = \log_2 \left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}} \right), \quad \text{and} \quad p_{uni} = \log_2 \left(\frac{E^N}{E^{2N}} \right).$$

In the numerical experiments, we have taken $\sigma_0 = 2$ for all the three test examples.

Example 5.1 *The first example is a linear two-point BVP studied by Stojanovic [14]*

$$-\varepsilon u''(x) + (1+x)^2 u(x) = [4x^2 - 14x + 4](1+x)^2, \quad x \in (0,1)$$
$$u(0) - u'(0) = 0, \quad u(1) = 0.$$

The maximum point–wise errors and rate of convergence are presented in Table 1 (see the Appendix).

The second test problem is the well-known example studied by several researchers subject to Dirichlet type boundary conditions, whereas here we have used mixed boundary conditions.

Example 5.2 *Consider the self–adjoint BVP*

$$-\varepsilon u''(x) + u(x) = -\cos^2(\pi x) - 2\varepsilon\pi^2 \cos(2\pi x), \quad x \in (0,1),$$
$$u(0) = 1, \quad 2u(1) + u'(1) = 1.$$

Table 2 (see the Appendix) corresponds to this example.

Example 5.3 *Here we consider the following semi–linear reaction–diffusion Robin problem*

$$-\varepsilon u_{xx} + (u^2 + u - 0.75)(u^2 + u - 3.75) = 0, \quad x \in (0,1),$$
$$u(0) = 1, \quad u(1) + u'(1) = 1.$$

To check the convergence of the nonlinear iteration, we took TOL $= 1.0e-08$. The maximum point–wise error and the rate of convergence are given in Table 3 (see the Appendix).

From the previous test examples one can easily notice that the present hybrid method produces accurate results. In general, it is difficult to obtain higher-order convergence results when the boundary conditions involve derivatives of the unknown. By the present method one can obtain second–order uniform convergent results which has been clearly seen from the results given in Tables 1-3 (see the Appendix).

6 Conclusions

To solve singularly perturbed reaction–diffusion Robin boundary-value problems numerically, we have devised an almost second–order (up to a logarithmic factor) uniformly convergent scheme, which is a proper combination of the classical finite difference scheme and the cubic spline scheme. The truncation error is derived, and it has been shown that the hybrid scheme is ε-uniform stable. Parameter-uniform error estimate is obtained for the present scheme. Two linear and one semilinear examples have tested for efficiency and accuracy of the method and they prove the theoretical estimates.

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Appendix

Table 1: *Maximum point-wise errors* E_ε^N*, rate of convergence* p *and* ε*-uniform errors* E^N*,* p_{uni} *for Example 5.1.*

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
2^0	2.2424e-3 2.0052	5.5859e-4 2.0015	1.3950e-4 2.0007	3.4859e-5 2.0003	8.7130e-6 2.0001	2.1781e-6 2.0002	5.4446e-7
2^{-2}	5.3790e-3 2.0109	1.3346e-3 2.0012	3.3338e-4 2.0007	8.3303e-5 2.0002	2.0823e-5 2.0000	5.2056e-6 2.0000	1.3014e-6
2^{-4}	2.0176e-2 2.0369	4.9167e-3 2.0092	1.2214e-3 2.0023	3.0487e-4 2.0006	7.6188e-5 2.0001	1.9045e-5 2.0000	4.7611e-6
2^{-6}	8.4393e-2 2.1413	1.9130e-2 2.0341	4.6708e-3 2.0084	1.1609e-3 2.0021	2.8980e-4 2.0005	7.2422e-5 2.0001	1.8104e-5
2^{-8}	3.2660e-1 2.0092	8.1128e-2 2.1362	1.8454e-2 2.0328	4.5100e-3 2.0081	1.1212e-3 2.0020	2.7990e-4 2.0005	6.9951e-5
2^{-10}	5.3911e-1 1.1372	2.4509e-1 1.6252	7.9449e-2 2.1336	1.8106e-2 2.0321	4.4269e-3 2.0079	1.1006e-3 2.0020	2.7478e-4
2^{-12}	5.2920e-1 1.1295	2.4189e-1 1.4974	8.5672e-2 1.7108	2.6171e-2 1.6265	8.4759e-3 1.6731	2.6578e-3 1.7004	8.1781e-4
2^{-14}	5.2430e-1 1.1256	2.4029e-1 1.4958	8.5203e-2 1.7103	2.6038e-2 1.6263	8.4345e-3 1.6730	2.6450e-3 1.7004	8.1388e-4
2^{-16}	5.2187e-1 1.1237	2.3949e-1 1.4950	8.4968e-2 1.7100	2.5972e-2 1.6261	8.4138e-3 1.6730	2.6386e-3 1.7004	8.1190e-4
2^{-18}	5.2065e-1 1.1228	2.3909e-1 1.4946	8.4850e-2 1.7098	2.5939e-2 1.6261	8.4034e-3 1.6730	2.6354e-3 1.7004	8.1091e-4
2^{-20}	5.2005e-1 1.1223	2.3889e-1 1.4944	8.4791e-2 1.7097	2.5922e-2 1.6260	8.3982e-3 1.6730	2.6337e-3 1.7004	8.1042e-4
2^{-24}	5.1959e-1 1.1219	2.3874e-1 1.4942	8.4747e-2 1.7097	2.5910e-2 1.6260	8.3943e-3 1.6730	2.6325e-3 1.7004	8.1005e-4
2^{-32}	5.1945e-1 1.1218	2.3869e-1 1.4942	8.4733e-2 1.7097	2.5906e-2 1.6260	8.3930e-3 1.6729	2.6322e-3 1.7004	8.0993e-4
2^{-36}	5.1944e-1 1.1218	2.3869e-1 1.4942	8.4732e-2 1.7097	2.5905e-2 1.6260	8.3930e-3 1.6729	2.6321e-3 1.7004	8.0993e-4
2^{-40}	5.1944e-1 1.1218	2.3869e-1 1.4942	8.4732e-2 1.7097	2.5905e-2 1.6260	8.3930e-3 1.6729	2.6321e-3 1.7004	8.0992e-4
E^N p_{uni}	5.3911e-1 1.1372	2.4509e-1 1.6252	8.5672e-2 1.6021	2.6171e-2 1.6265	8.4759e-3 1.6731	2.6578e-3 1.7004	8.1781e-4

Table 2: *Maximum point-wise errors* E_ε^N*, rate of convergence* p *and* ε*-uniform errors* E^N*,* p_{uni} *for Example 5.2.*

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
2^0	8.5625e-3 1.9899	2.1557e-3 1.9975	5.3985e-4 1.9994	1.3502e-4 1.9997	3.3763e-5 2.0000	8.4409e-6 2.0000	2.1103e-6
2^{-2}	7.1029e-3 1.9914	1.7863e-3 1.9979	4.4722e-4 1.9986	1.1192e-4 1.9999	2.7982e-5 2.0000	6.9956e-6 1.9999	1.7490e-6
2^{-4}	5.4375e-3 1.9929	1.3661e-3 1.9968	3.4228e-4 1.9998	8.5582e-5 1.9998	2.1399e-5 2.0000	5.3498e-6 2.0000	1.3375e-6
2^{-6}	5.2979e-3 2.0331	1.2944e-3 2.0041	3.2269e-4 2.0020	8.0564e-5 2.0001	2.0140e-5 2.0001	5.0345e-6 2.0000	1.2586e-6
2^{-8}	2.5516e-2 2.1307	5.8265e-3 2.0317	1.4250e-3 2.0079	3.5430e-4 2.0020	8.8456e-5 2.0005	2.2106e-5 2.0001	5.5262e-6
2^{-10}	5.1693e-2 1.4660	1.8712e-2 1.6627	5.9099e-3 2.0314	1.4457e-3 2.0078	3.5947e-4 2.0019	8.9747e-5 2.0005	2.2429e-5
2^{-12}	5.1752e-2 1.4662	1.8730e-2 1.5483	6.4041e-3 1.5895	2.1280e-3 1.6202	6.9220e-4 1.6628	2.1861e-4 1.6971	6.7420e-5
2^{-14}	5.1743e-2 1.4659	1.8732e-2 1.5483	6.4046e-3 1.5895	2.1281e-3 1.6202	6.9225e-4 1.6629	2.1862e-4 1.6971	6.7425e-5
2^{-16}	5.1728e-2 1.4655	1.8731e-2 1.5483	6.4047e-3 1.5895	2.1281e-3 1.6202	6.9225e-4 1.6629	2.1862e-4 1.6971	6.7425e-5
2^{-18}	5.1718e-2 1.4652	1.8731e-2 1.5482	6.4047e-3 1.5895	2.1281e-3 1.6202	6.9225e-4 1.6629	2.1862e-4 1.6971	6.7425e-5
2^{-20}	5.1712e-2 1.4651	1.8731e-2 1.5482	6.4046e-3 1.5895	2.1281e-3 1.6202	6.9225e-4 1.6629	2.1862e-4 1.6971	6.7425e-5
2^{-24}	5.1707e-2 1.4650	1.8731e-2 1.5482	6.4046e-3 1.5895	2.1281e-3 1.6202	6.9225e-4 1.6629	2.1862e-4 1.6971	6.7425e-5
2^{-28}	5.1706e-2 1.4650	1.8730e-2 1.5482	6.4046e-3 1.5895	2.1281e-3 1.6202	6.9225e-4 1.6629	2.1862e-4 1.6971	6.7425e-5
2^{-32}	5.1706e-2 1.4649	1.8730e-2 1.5482	6.4046e-3 1.5895	2.1281e-3 1.6202	6.9225e-4 1.6629	2.1862e-4 1.6971	6.7425e-5
E^N p_{uni}	5.1752e-2 1.4661	1.8732e-2 1.5483	6.4047e-3 1.5895	2.1281e-3 1.6202	6.9225e-4 1.6629	2.1862e-4 1.6971	6.7425e-5

Table 3: *Maximum point-wise errors E_ε^N, rate of convergence p and ε-uniform errors E^N, p_{uni} for Example 5.3.*

ε	Number of mesh points N				
	64	128	256	512	1024
2^0	9.6742e-6 1.9998	2.4189e-6 2.0000	6.0472e-7 2.0000	1.5118e-7 2.0000	3.7794e-8
2^{-2}	3.1377e-4 1.9971	7.8598e-5 1.9987	1.9667e-5 1.9998	4.9173e-6 2.0000	1.2294e-6
2^{-4}	1.2969e-3 1.9889	3.2673e-4 1.9973	8.1839e-5 1.9987	2.0479e-5 1.9998	5.1203e-6
2^{-6}	4.8940e-3 1.9154	1.2974e-3 1.9889	3.2685e-4 1.9973	8.1868e-5 1.9987	2.0486e-5
2^{-8}	1.7322e-2 1.8235	4.8940e-3 1.9154	1.2974e-3 1.9889	3.2685e-4 1.9973	8.1868e-5
2^{-10}	3.6848e-2 1.0890	1.7322e-2 1.8235	4.8940e-3 1.9154	1.2974e-3 1.9889	3.2685e-4
2^{-12}	3.9932e-2 0.9206	2.1096e-2 1.2395	8.9346e-3 1.5457	3.0603e-3 1.6476	9.7672e-4
2^{-14}	3.9932e-2 0.9206	2.1096e-2 1.2395	8.9346e-3 1.5457	3.0603e-3 1.6476	9.7672e-4
2^{-16}	3.9932e-2 0.9206	2.1096e-2 1.2395	8.9346e-3 1.5457	3.0603e-3 1.6476	9.7672e-4
2^{-18}	3.9932e-2 0.9206	2.1096e-2 1.2395	8.9346e-3 1.5457	3.0603e-3 1.6476	9.7672e-4
2^{-20}	3.9932e-2 0.9206	2.1096e-2 1.2395	8.9346e-3 1.5457	3.0603e-3 1.6476	9.7672e-4
2^{-24}	3.9932e-2 0.9206	2.1096e-2 1.2395	8.9346e-3 1.5457	3.0603e-3 1.6476	9.7672e-4
2^{-32}	3.9932e-2 0.9206	2.1096e-2 1.2395	8.9346e-3 1.5457	3.0603e-3 1.6476	9.7672e-4
E^N p_{uni}	3.9932e-2 0.9206	2.1096e-2 1.2395	8.9346e-3 1.5457	3.0603e-3 1.6476	9.7672e-4