European Society of Computational Methods in Sciences and Engineering (ESCMSE)

Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM)
vol. 2, no. 1-2, 2007, pp. 1-9
ISSN 1790–8140

Stability Analysis of Linear Multistep Methods via Polynomial Type Variation[1]

L. Aceto [2]

Dipartimento di Matematica Applicata "U. Dini",
Università di Pisa,
Via Buonarroti 1, I-56127 Pisa

R. Pandolfi

Dipartimento di Matematica "U. Dini",
Università di Firenze,
Viale Morgagni 67/A, I-50134 Firenze

D. Trigiante

Dipartimento di Energetica "S. Stecco",
Università di Firenze,
Via C. Lombroso 6/17, I-50134 Firenze

Received 20 May, 2006; accepted in revised form 8 September, 2006

Abstract: The linear stability analysis for linear multistep methods leads to study the location of the roots of the associated characteristic polynomial with respect to the unit circle in the complex plane. It is known that if the discrete problem is an initial value one, it is sufficient to determine when all the roots are inside the unit disk. This requirement is, however, conflicting with the order conditions, as established by the Dahlquist barrier. The conflict disappears if one uses a linear multistep method coupled with boundary conditions (BVMs). In this paper, a rigorous analysis of the linear stability for some classes of BVMs is presented. The study is carried out by using the notion of *type* of a polynomial.

Keywords: Linear multistep methods, Stability of numerical methods, polynomial type

Mathematics Subject Classification: 65L06, 65L20

1 Introduction

One of the most important problems to be solved when a first-order differential equation in $\mathbb{R}^s$ is approximated by a linear multistep method is the control of the errors between the continuous and the discrete solutions. In the last forty years a lot of efforts have been done in this field, mainly when the methods are applied to dissipative problems. In such a case, the study of the propagation of the errors is made by means of a linear difference equation (frequently called *error equation*)

[1]Published electronically April 14, 2007
[2]Corresponding author. E-mail: l.aceto@dma.unipi.it

depending on a complex parameter. Considering that the resulting equation is, in general, of order greater than one, some additional conditions must be fixed in order to get the solution we are interested in. When all of them are chosen at the first points of the interval of integration we are solving discrete initial value methods (IVMs). It is well-known that in this case the asymptotic stability of the zero solution of the error equation is equivalent to require that the associate stability polynomial is a Schur polynomial, that is all its roots have modulus less than one, as the complex parameter, say q, varies into the left-hand complex plane. This request is, however, conflicting with the order conditions, as established by the Dahlquist barrier. On the other hand, since only one condition is inherited by the continuous problem there is no valid reason to use discrete IVMs. It is possible to split the additional conditions part at the beginning and part at the end of the interval of integration, solving a boundary value method (BVM). In this case the concept of stability needs to be generalized. In such a case the notion of well-conditioning is more appropriate. Essentially, such notion requires that under a perturbation $\delta\eta$ of the imposed conditions, the perturbation of the solution δy should be bounded as follows

$$\|\delta y\| \leq \kappa \|\delta\eta\|$$

where κ is independent on the number of points in the discrete mesh. If a discrete boundary value problem is considered, the error equation is well-conditioned if the number of initial conditions is equal to the number of roots of the stability polynomial inside the unit circle and the number of conditions at the end of the interval of integration is equal to the number of roots outside of the unit circle [6]. This result generalizes the stability condition for IVMs where all the roots need to be inside the unit disk. In order to control that the number of roots inside the unit circle is constant for $q \in \mathbb{C}^-$, the notion of *type* of a polynomial $p(z)$ of degree k is useful. A polynomial is of type:

$$\mathsf{T}(p(z)) = (r_1, r_2, r_3), \qquad r_1 + r_2 + r_3 = k,$$

if r_1 is the number of roots inside, r_2 on the boundary and r_3 outside the unit disk in the complex plane, respectively.

Denoting, as usual, by $\rho_k(z)$ and $\sigma_k(z)$ the first and second characteristic polynomials of a generic k-step linear multistep method and by $q = h\lambda$ the characteristic complex parameter, where h is the stepsize and λ is the parameter in the usual test equation (i.e., $y' = \lambda y$), the classical A-stability condition requires that $\mathsf{T}(\rho_k(z) - q\,\sigma_k(z)) = (k, 0, 0)$ for all $q \in \mathbb{C}^-$. In the BVM approach, the A-stability condition requires that only r_2 vanishes, i.e.,

$$\mathsf{T}(\rho_k(z) - q\,\sigma_k(z)) = (r_1, 0, r_3), \qquad \text{for all } q \in \mathbb{C}^-.$$

The stability problem is then reduced to study whenever or not $\mathsf{T}(\rho_k(z) - q\,\sigma_k(z))$ remains constant for $q \in \mathbb{C}^-$, no matter if the discrete problem is an IVM or a BVM one. Of course, we have now much more freedom since the case $r_3 = 0$ is only one of the allowed possibilities.

Except for very simple cases, the proof that the number of roots inside the unit circle remains constant for all $q \in \mathbb{C}^-$ is often checked numerically. In this paper a theoretical analysis is carried out for the classes of linear multistep methods generalizing the Backward Differentiation Formulas (GBDFs) and the Adams methods (OGAMs, GAMs) [2, 6]. The starting point of this study will be either the explicit form of the coefficients or relevant properties of them.

2 Polynomial type and the stability problem for LMMs

The study of the location of the zeros of special polynomials with respect to a curve in the complex plane is an old problem whose pioneering works go back to Schur [7]. Classical criteria such as Schur's or Routh-Hurwitz are in general difficult to apply to high degree polynomials. The following result provide conditions which are simpler to check in our subsequent analysis.

Theorem 2.1 *Let $p(z)$ be the real polynomial of degree k defined by*

$$p(z) = \sum_{j=0}^{k} a_j z^j$$

and $p^*(z) = z^k\, p(z^{-1})$ its adjoint. Suppose that the following conditions are satisfied:

i) $p(1) \neq 0$;

ii) there exists $m \in \mathbb{N}$, $m \leq k$, such that:

$$z^{2\,m-k+1} p(z) - p^*(z) = a_k (z-1)^{2\,m+1}.$$

Then,

$$T(p(z)) = \begin{cases} (k-m, 0, m) & \text{if } (-1)^m\, a_k\, p(1) > 0, \\ (k-m-1, 0, m+1) & \text{if } (-1)^m\, a_k\, p(1) < 0. \end{cases}$$

Proof. We first prove that $p(z)$ has no zeros on the unit circle when $p(1) \neq 0$. Suppose that $p(e^{i\hat{\theta}}) = 0$, for a fixed $\hat{\theta} \in (0, 2\pi)$. Then, $p^*(e^{i\hat{\theta}}) = e^{ik\hat{\theta}} p(e^{-i\hat{\theta}}) = 0$ since the coefficients are real. Moreover, from hypothesis *ii*) it turns out to be

$$0 = e^{i(2m-k+1)\hat{\theta}} p(e^{i\hat{\theta}}) - p^*(e^{i\hat{\theta}}) = a_k (e^{i\hat{\theta}} - 1)^{2m+1}.$$

This equality is only verified for $\hat{\theta} = 0$, which is excluded by hypothesis *i*).
Let g be defined by

$$g(z) = \frac{z^{2(m-k)+1}\, p(z^2)}{p(1)}.$$

Supposing that n is the number of zeros of p inside the unit circle (counted with multiplicity), g has $n_p \equiv 2(k-m) - 1$ poles and $n_r \equiv 2n$ zeros inside the unit circle. We determine the winding number w of $g(z)$ around the circle $|z| = 1$ (see, e.g., [7]). The hypothesis *ii*), relating p and p^*, yields

$$\begin{aligned} g(z) - g(z^{-1}) &= \frac{z^{2(2m-k+1)}\, p(z^2) - z^{2k}\, p(z^{-2})}{z^{2m+1}\, p(1)} = \frac{z^{2(2m-k+1)}\, p(z^2) - p^*(z^2)}{z^{2m+1}\, p(1)} \\ &= \frac{a_k (z^2-1)^{2m+1}}{z^{2m+1}\, p(1)} = \frac{a_k (z - z^{-1})^{2m+1}}{p(1)}. \end{aligned}$$

Then,

$$\begin{aligned} \text{Im}\; g(e^{i\theta}) &= \frac{g(e^{i\theta}) - g(e^{-i\theta})}{2\,i} = \frac{a_k (e^{i\theta} - e^{-i\theta})^{2m+1}}{2\,i\,p(1)} = \frac{a_k (2\,i\,\sin\theta)^{2m+1}}{2\,i\,p(1)} \\ &= \frac{(-1)^m a_k}{2\,p(1)} (2\,\sin\theta)^{2m+1}. \end{aligned}$$

This quantity vanishes for $\theta = 0, \pi$. Moreover, according to the sign of $(-1)^m\, a_k\, p(1)$ it is either positive or negative between $(0, \pi)$ and the opposite between $(\pi, 2\pi)$. Considering that w assumes the value $+1$ in the first case and -1 in the other one and that the principle of the argument yields $w = n_r - n_p = 2(n-k+m) + 1$, immediately it follows that $n = k - m$, if $(-1)^m\, a_k\, p(1) > 0$ and $n = k-m-1$, if $(-1)^m\, a_k\, p(1) < 0$. Since p has a total of k zeros, the assertion follows. ∎

Example 2.1 *Let*

$$p(z) = 10 - 5z + z^2.$$

It satisfies the hypothesis ii) with $m = k = 2$. *Moreover, since* $(-1)^m\, a_k\, p(1) = 6$, *it follows that* $\mathcal{T}(p(z)) = (0, 0, 2)$. *In fact, the roots of* $p(z)$ *are:* $z_1 = \frac{5+i\sqrt{15}}{2}, z_2 = \frac{5-i\sqrt{15}}{2}$.

Example 2.2 *Consider the polynomial*

$$p(z) = -7 + 3z + 5z^2 + z^3.$$

In this case the hypothesis ii) is verified with $m = 2$. *Moreover, since* $(-1)^m\, a_k\, p(1) = 2$, *we get* $\mathcal{T}(p(z)) = (1, 0, 2)$. *In fact, the roots of* $p(z)$ *are:* $z_1 \cong 0.8662, z_2 \cong -2.2108, z_3 \cong -3.6554$.

Example 2.3 *Let*

$$p(z) = -9 + z + 5z^2 + z^3.$$

Here $p(1) = -2$ *and the condition ii) is verified with* $m = 2$. *Since* $(-1)^m\, a_k\, p(1) = -2$, *one has that* $\mathcal{T}(p(z)) = (0, 0, 3)$. *The roots are:* $z_1 \cong -4.2731, z_2 \cong -1.8596, z_3 \cong 1.1326$.

The above result turns out to be useful in studying the stability properties for discrete problems obtained by using *Boundary Value Methods* (BVMs) with (k_1, k_2)-*boundary conditions*, i.e., k-step linear multistep methods (LMMs) to which k_1 initial conditions and $k_2 = k - k_1$ final ones are imposed.
For the sake of completeness, we briefly report the essential results on the stability problem for BVMs (see [6] for details). The *characteristic* (or *stability*) *polynomial* of the method is

$$\pi_k(z, q) = \rho_k(z) - q\sigma_k(z), \quad q \in \mathbb{C}, \tag{1}$$

where

$$\rho_k(z) := \sum_{j=0}^{k} \alpha_j^{(k)} z^j, \qquad \sigma_k(z) := \sum_{j=0}^{k} \beta_j^{(k)} z^j. \tag{2}$$

Although the locution "well-conditioning" would be more appropriate dealing with boundary value problems, we shall continue to use the term stability, as customary. It is known that the well-conditioning of a linear boundary value problem, either continuous or discrete, is related to the so called *dichotomy*. In the discrete case it essentially states that the number of initial conditions should be equal to the number of roots of the characteristic polynomial inside the unit circle and, of course, the number of conditions at the end of the interval of integration should be equal to the number of roots outside the unit circle. Then, supposing that k_1 is the number of the initial conditions and k_2 the number of the final ones, the generalization of the stability request to boundary value problems can be done as follows.

Definition 2.1 *The* k*-step BVM with* (k_1, k_2)*-boundary conditions is said to be* A_{k_1,k_2}-stable *if* $\mathbb{C}^- \subseteq \mathcal{D}_{k_1,\, k_2}$, *where*

$$\mathcal{D}_{k_1,\, k_2} = \{q \in \mathbb{C} : \mathcal{T}(\pi_k(z, q)) = (k_1, 0, k_2)\}$$

denotes the region of (k_1, k_2)*-absolute stability.*

When $k_2 = 0$ the above definition reduces to the classical A-stability.
The notion of consistency remains unchanged with respect to the IVM case, i.e.,

$$\rho_k(1) = 0, \qquad \rho_k'(1) = \sigma_k(1). \tag{3}$$

We choose here the normalization condition $\sigma_k(1) = 1$.

Theorem 2.2 *Consider a consistent k-step BVM with (k_1, k_2)-boundary conditions $(k_1 + k_2 = k)$ and let $\pi_k(z, q)$ be the characteristic polynomial defined by (1)-(2). Suppose that*

i) $\mathsf{T}(\rho_k(z)) = (k_1 - 1, 1, k_2)$;

ii) $\sigma_k(e^{i\theta}) \neq 0, \quad \forall\, \theta \in [0, 2\pi)$;

iii) $\mathrm{Re}\left(\dfrac{\rho_k(e^{i\theta})}{\sigma_k(e^{i\theta})}\right) > 0, \quad \forall\, \theta \in (0, 2\pi)$.

Then the method is A_{k_1,k_2}-stable.

Proof. The roots of the characteristic polynomial $\pi_k(z, q)$ will depend continuously on the complex parameter q. Denoting by $z(q)$ one of them, we get (see (1))

$$\rho_k(z(q)) - q\sigma_k(z(q)) = 0.$$

By differentiating the above relation with respect to q we obtain

$$\rho_k'(z(q))\,\frac{dz}{dq} - q\,\sigma_k'(z(q))\,\frac{dz}{dq} - \sigma_k(z(q)) = 0$$

or, equivalently,

$$\frac{dz}{dq} = \frac{\sigma_k(z(q))}{\rho_k'(z(q)) - q\,\sigma_k'(z(q))}.$$

From hypothesis *i)* and the consistency conditions the root on the unit circle is $\tilde{z}(0) = 1$. We show first that for "sufficiently small" values of $q \in \mathbb{C}^-$, one has that $\mathsf{T}(\pi_k(z, q)) = (k_1, 0, k_2)$. In fact, in a neighborhood of $q = 0$, say $\mathcal{B}$, which is such that the roots strictly inside the unit circle remain there and similarly for those outside of it, concerning the trajectory of the root on the unit circle for $q = 0$, we have

$$\tilde{z}(q) = \tilde{z}(0) + \left.\frac{d\tilde{z}}{dq}\right|_{q=0} q + O(q^2) = 1 + q + O(q^2).$$

Consequently, as q varies in $\mathcal{B} \cap \mathbb{C}^-$, the root $\tilde{z}(q)$ enters the unit circle. Then, for such values of q, $\mathsf{T}(\pi_k(z, q)) = (k_1, 0, k_2)$.
Suppose now that there exists $\hat{q} \in \mathbb{C}^-$ such that a generic root $z(\hat{q})$ of the characteristic polynomial crosses the unit circle, i.e.,

$$\pi_k(z(\hat{q}), \hat{q}) \equiv \pi_k(e^{i\hat{\theta}}, \hat{q}) = 0, \qquad \hat{\theta} \in (0, 2\pi).$$

Considering that this is equivalent to

$$\hat{q} = \frac{\rho_k(e^{i\hat{\theta}})}{\sigma_k(e^{i\hat{\theta}})},$$

from hypothesis *iii)* it follows that $\mathrm{Re}(\hat{q}) > 0$. But this is conflicting with the assumption $\hat{q} \in \mathbb{C}^-$. Consequently, as q varies in the negative half complex plane, $\pi_k(z, q)$ does not change its type, that is it remains $\mathsf{T}(\pi_k(z, q)) = (k_1, 0, k_2)$. This implies the assertion. ■

3 Some relevant applications

The above results are now applied to determine the type of polynomials associated to some classes of linear multistep formulae, used as BVMs. In particular, the considered methods have a priori fixed either $\rho_k(z)$ or $\sigma_k(z)$. The coefficients of the unknown polynomial are determined by imposing that the method has the highest possible order.

3.1 Generalized BDF

The methods in the class of the *Generalized Backward Differentiation Formulas* (GBDFs) [6] are characterized to have a priori fixed the polynomial

$$\sigma_k(z) = z^{\nu}, \tag{4}$$

with

$$\nu = \begin{cases} \dfrac{k+2}{2} & \text{for even } k \\[2ex] \dfrac{k+1}{2} & \text{for odd } k \end{cases} . \tag{5}$$

By imposing the highest possible order, that is k, in [4] the explicit form of the coefficients characterizing the first characteristic polynomial $\rho_k(z)$ was obtained:

$$\alpha_j^{(k)} = \begin{cases} \displaystyle\sum_{r=1}^{j} \frac{1}{r} - \sum_{r=1}^{k-j} \frac{1}{r}, & \text{if } j = \nu \\[3ex] \dfrac{(-1)^{\nu-j}}{\nu - j} \dfrac{\binom{k}{j}}{\binom{k}{\nu}}, & \text{if } j \neq \nu \end{cases} \qquad j = 0, 1, \ldots, k. \tag{6}$$

Such coefficients leads to assert that the polynomials $\rho_k(z)$ satisfy the following relation [5]:

$$\rho_k(z) + z^{2\nu} \rho_k(z^{-1}) = \frac{(-1)^{\nu}}{\nu \binom{k}{\nu}} (1 - z)^{2\nu}, \tag{7}$$

with ν given by (5). From (3) and the normalization condition we get

$$\frac{\rho_k(z)}{z-1} = \psi_{k-1}(z) := \sum_{j=0}^{k-1} a_j^{(k-1)} z^j, \qquad \psi_{k-1}(1) = \rho_k'(1) = 1. \tag{8}$$

By using (6) it is easy to check that

$$a_0^{(k-1)} = -\alpha_0^{(k)} \equiv \frac{(-1)^{\nu-1}}{\nu \binom{k}{\nu}}. \tag{9}$$

In order to establish the type of $\rho_k(z)$ it is sufficient to determine the type of $\psi_{k-1}(z)$, or, equivalently, the type of

$$\psi_{k-1}^*(z) = z^{k-1} \psi_{k-1}(z^{-1}). \tag{10}$$

Theorem 3.1 *Let $\psi^*_{k-1}(z)$ be the polynomial defined in* (10) *and ν given by* (5). *Then,* $\mathsf{T}(\psi^*_{k-1}(z)) = (k-\nu, 0, \nu-1)$.

Proof. We only consider here the case k odd, i.e., $k = 2\nu - 1$, since the case with k even proceeds similarly.
From (8) and (10) it follows that $\psi^*_{k-1}(1) = 1$. Moreover, by using (7) and (8) we get

$$-\psi_{k-1}(z) + z^{2\nu-1}\psi_{k-1}(z^{-1}) = \frac{(-1)^\nu}{\nu \begin{pmatrix} k \\ \nu \end{pmatrix}}(1-z)^{2\nu-1},$$

which, taking into account (9) and (10), leads to

$$z\psi^*_{k-1}(z) - \psi_{k-1}(z) = \frac{(-1)^{\nu-1}}{\nu \begin{pmatrix} k \\ \nu \end{pmatrix}}(z-1)^{2\nu-1} = a_0^{(k-1)}(z-1)^{2\nu-1}.$$

Considering that the quantity

$$\frac{(-1)^{\nu-1}\psi^*_{k-1}(1)}{a_0^{(k-1)}} = \nu \begin{pmatrix} k \\ \nu \end{pmatrix}$$

is positive, by applying Theorem 2.1 the thesis follows. ∎

Corollary 3.1 *Let $\rho_k(z)$ be the polynomial associated to the k-step GBDF and ν given by* (5). *Then,* $\mathsf{T}(\rho_k(z)) = (\nu-1, 1, k-\nu)$.

Proof. The above theorem and (10) imply that $\mathsf{T}(\psi_{k-1}(z)) = \mathsf{T}(\psi^*_{k-1}(z^{-1})) = (\nu-1, 0, k-\nu)$. From (8) the thesis follows. ∎

Finally, we are in the position to get the stability result.

Theorem 3.2 *The k-step GBDFs are $A_{\nu,k-\nu}$-stable.*

Proof. Thanks to Corollary 3.1 and (4) the first two hypotheses of Theorem 2.2 are verified. Concerning hypothesis *iii*), by using (4) and (7) we have

$$\begin{aligned}
\mathrm{Re}\left(\frac{\rho_k(e^{i\theta})}{\sigma_k(e^{i\theta})}\right) &= \mathrm{Re}\left(\frac{\rho_k(e^{i\theta})}{e^{i\nu\theta}}\right) = \frac{1}{2}\frac{\rho_k(e^{i\theta}) + e^{i2\nu\theta}\rho_k(e^{-i\theta})}{e^{i\nu\theta}} \\
&= \frac{(-1)^\nu}{2\nu \begin{pmatrix} k \\ \nu \end{pmatrix}} \frac{(1-e^{i\theta})^{2\nu}}{e^{i\nu\theta}} = \frac{(-1)^\nu}{2\nu \begin{pmatrix} k \\ \nu \end{pmatrix}} \left(-2i\sin\frac{\theta}{2}\right)^{2\nu} \\
&= \frac{2^{2\nu-1}}{\nu \begin{pmatrix} k \\ \nu \end{pmatrix}} \left(\sin\frac{\theta}{2}\right)^{2\nu} > 0, \qquad \forall\,\theta \in (0, 2\pi).
\end{aligned}$$

Therefore, the thesis follows. ∎

3.2 Generalizations of the Adams methods: GAMs and OGAMs

Consider now a class of methods generalizing the classical Adams methods. They are characterized to have the first characteristic polynomial a priori fixed as follows:

$$\rho_k(z) = z^{\ell-1}(z-1), \tag{11}$$

with

$$\ell = \begin{cases} \nu & \text{for } k = 2\nu \\ \nu - 1 & \text{for } k = 2\nu - 1 \end{cases} \qquad k > 1. \tag{12}$$

In particular, they are called *Generalized Adams Methods* (GAMs) when $k = 2\nu$ and *Odd-Generalized Adams Methods* (OGAMs) when $k = 2\nu - 1$ [2, 6]. All of them have order $k+1$.
To study the stability problem for this family of methods the knowledge of the explicit form of the coefficients $\beta_j^{(k)}$ characterizing the polynomial $\sigma_k(z)$ is not necessary. In fact, in next theorem only the following properties will be used:

1) the sign of the coefficient $\beta_k^{(k)}$ satisfies the following relation:

$$\text{sign}\left(\beta_k^{(k)}\right) = (-1)^{k-\ell} \quad \text{for } k > 1, \tag{13}$$

where ℓ is given in (12);

2) for $k > 1$ ($k = 2\nu, 2\nu - 1$) one has that

$$z^{2\nu-k+1}\sigma_k(z) - \sigma_k^*(z) = \beta_k^{(k)}(z-1)^{2\nu+1}, \tag{14}$$

where $\sigma_k^*(z)$ denotes the adjoint polynomial of $\sigma_k(z)$.

In the GAMs case the proof of the previous properties can be found in [3]. Such proof can be easily extended to the OGAMs case (see [1] for more details).

Theorem 3.3 *Let ℓ be defined in* (12)*. Then,*

1. *the k-step GAMs are $A_{\ell,\ell}$-stable, for k even;*

2. *the k-step OGAMs are $A_{\ell,\ell+1}$-stable, for k odd.*

Proof. Consider the case of k even, i.e., $k = 2\nu \equiv 2\ell$. Trivially, from (11) $\mathsf{T}(\rho_k(z)) = (\ell-1, 1, \ell)$. Moreover, by using (14) and the normalization condition $\sigma_k(1) = 1$, Theorem 2.1 implies that $\mathsf{T}(\sigma_k(z)) = (\ell, 0, \ell)$. Consequently, the conditions *i*) and *ii*) of Theorem 2.2 are both satisfied. We only need to check that hypothesis *iii*) holds true. From (11) one has

$$\begin{aligned}
\text{Re}\left(\frac{\rho_k(e^{i\theta})}{\sigma_k(e^{i\theta})}\right) &= \text{Re}\left(\frac{e^{i(\ell-1)\theta}(e^{i\theta}-1)}{\sigma_k(e^{i\theta})}\right) \\
&= \frac{1}{2}\left(\frac{e^{i(\ell-1)\theta}(e^{i\theta}-1)}{\sigma_k(e^{i\theta})} + \frac{e^{-i(\ell-1)\theta}(e^{-i\theta}-1)}{\sigma_k(e^{-i\theta})}\right) \\
&= \frac{e^{i\ell\theta}(1-e^{-i\theta})\sigma_k(e^{-i\theta}) - e^{-i\ell\theta}(1-e^{-i\theta})e^{i\theta}\sigma_k(e^{i\theta})}{2|\sigma_k(e^{i\theta})|^2} \\
&= \frac{e^{-i\ell\theta}(1-e^{-i\theta})}{2|\sigma_k(e^{i\theta})|^2}\left(e^{i2\ell\theta}\sigma_k(e^{-i\theta}) - e^{i\theta}\sigma_k(e^{i\theta})\right).
\end{aligned}$$

By using (14) the previous relation can be recast as

$$\mathrm{Re}\left(\frac{\rho_k(e^{i\theta})}{\sigma_k(e^{i\theta})}\right) = \frac{\beta_k^{(k)}(e^{-i\theta}-1)e^{-i\ell\theta}(e^{i\theta}-1)^{2\ell+1}}{2|\sigma_k(e^{i\theta})|^2} = \frac{(-1)^\ell\beta_k^{(k)}2^\ell(1-\cos\theta)^{\ell+1}}{|\sigma_k(e^{i\theta})|^2}.$$

Consequently, by taking into account (13) it turns out that

$$\mathrm{Re}\left(\frac{\rho_k(e^{i\theta})}{\sigma_k(e^{i\theta})}\right) > 0, \quad \forall\,\theta \in (0, 2\pi).$$

Therefore, the thesis follows for the GAMs case.
The proof for the OGAMs case (k odd) proceeds similarly with only minor changes and it is omitted for brevity. ∎

4 Conclusions

A rigorous analysis of the linear stability for some classes of BVMs has been made by using information on the variation of *type* of polynomials associated to the methods. We have proved that all the considered k-step methods are A-stable, in the generalized sense, of order either k or $k+1$. Then they are suitable for approximating the solutions of stiff problems. As matter of fact, the GAM code, dealing with stiff IVPs, is based on some methods in the GAMs class [8].

Acknowledgment

The authors would like to thank the anonymous referees for their comments and suggestions.

References

[1] L. Aceto, Properties of the Pascal matrix and their application in the stability problem for the generalized Adams methods, *Tech. Report, 11/2002, Department of Mathematics, University of Florence, I,* 2002.

[2] L. Aceto, Some applications of the Pascal matrix to the study of numerical methods for differential equations, *Atti del XVII Congresso U.M.I.*. 395-407(2004).

[3] L. Aceto, R. Pandolfi and D. Trigiante, Theoretical analysis of the stability for Extended Trapezoidal Rules (submitted).

[4] L. Aceto and D. Trigiante, On the A-stable methods in the GBDF class, *Nonlinear Analysis: Real World Applications*, **3** 9-23(2002).

[5] L. Aceto and D. Trigiante, On the properties of matrices defining some classes of BVMs, *Numerische Mathematik*, **96** 1-16(2003).

[6] L. Brugnano and D. Trigiante: *Solving differential problems by multistep initial and boundary value methods.* Gordon and Breach Science Publishers, Amsterdam, 1998.

[7] P. Henrici: *Applied and computational complex analysis.* Vol. 1, John Wiley & Sons, New York, 1974.

[8] F. Iavernaro, F. Mazzia, *GAM code.* Available online at: http://www.dm.uniba.it/∼mazzia/ode/readme.html