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Numerical Solution of Stochastic Differential Equations with Additive Noise by Runge–Kutta Methods[1]

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Abstract: In this paper we study the numerical treatment of Stochastic Differential Equations with additive noise and one dimensional Wiener process. We develop two, three and four stage Runge–Kutta methods which attain deterministic order up to four and stochastic order up to one and a half specially constructed for this class of problems. Numerical tests and comparisons with other known methods in the solution of various problems justify our effort, especially for our three stages methods.

Keywords: Stochastic Differential Equations, Additive Noise, Numerical Solution, Runge–Kutta methods

Mathematics Subject Classification: 65L06, 65C30

1 Introduction

Stochastic Differential Equations (SDEs) are used as models in a range of fields such as mechanics, biology, chemistry, epidemiology, finance etc [1, 2]. One–Wiener process Stratonovich autonomous SDE is given by

$$dy(t) = f(y(t))dt + g(y(t)) \circ dW(t),\ t \geq t_0, \tag{1}$$

which in an integral form can written as,

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y(s))ds + \int_{t_0}^{t} g(y(s)) \circ dW(s). \tag{2}$$

Just as in the deterministic differential equations any SDE can be written in an autonomous form if we add another component which represents time. In this work we focus on a subclass of (1) called SDEs with additive noise, where g is constant. In this case the Ito and Stratonovich

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forms of the SDE are the same. Such equations arise in many applications and in integral form can be written as follows,

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y(s))ds + \int_{t_0}^{t} \eta \circ dW(s). \tag{3}$$

We are interested in the numerical treatment of (3) using Runge–Kutta (RK) methods. Pamela Burrage [4] in her PhD thesis has introduced the usage of RK methods for the solution of general problem (2). Motivated by her work and the work of Xanthos and Papageorgiou [5], we present classes of RK methods especially constructed for SDEs with additive noise. In the following section we discuss the numerical solution of SDEs by RK methods. In Section 3 we present classes of RK methods methods specially constructed for SDEs with additive noise. Finally, in the last section we form our numerical tests.

2 RK methods for SDEs

The general form of a s-stage stochastic Runge-Kutta method for the one Wiener process is given by

$$k_i = y_n + \sum_{j=1}^{s} Z_{ij}^{(0)} f(k_j) + \sum_{j=1}^{s} Z_{ij}^{(1)} g(k_j),\ i = 1, 2, \cdots, s$$

$$y_{n+1} = y_n + \sum_{j=1}^{s} z_j^{(0)} f(k_j) + \sum_{j=1}^{s} z_j^{(1)} g(k_j). \tag{4}$$

In (4) matrices $Z^{(0)} = (Z_{ij}^{(0)}), Z^{(1)} = (Z_{ij}^{(1)})$ and vectors $z^{(0)} = (z_j^{(0)}), z^{(1)} = (z_j^{(1)})$ have the form

$$Z^{(0)} = hA, z^{(0)} = h\alpha, Z^{(1)} = \sum_{l=1}^{p} B^{(l)}\theta_l, z^{(1)} = \sum_{l=1}^{p} \gamma^{(l)}\theta_l$$

where $A = (A_{ij})$ and $B^{(l)} = (B_{ij}^{(l)})$ for $l = 1, 2, \cdots, p$ are $s \times s$ real matrices and $\alpha = (\alpha_j)$ and $\gamma^{(l)} = (\gamma_j^{(l)})$ are $s \times 1$ row vectors with real elements and h is the step of the method. If A and $B^{(l)}\,'s$ are strictly lower triangular, then the RK method is said to be explicit and its implementation is straight forward. For simplifying purposes we assume that elements of $Z^{(1)}$ and $z^{(1)}$ are written as linear combinations of the random variables $\theta_l, l = 1, \ldots, p$. These variables are called stochastic steps of the method and follow a distribution which arises from multiple stochastic integrals such as $J_{\{1\}}$ and $J_{\{1,0\}}$ (see Burrage [4]). Then, an explicit stochastic RK method has the form:

$$k_i = y_n + h\sum_{j=1}^{i-1} A_{ij} f(k_j) + \sum_{l=1}^{p}(\sum_{j=1}^{i-1} B_{ij}^{(l)} g(k_j))\theta_l$$

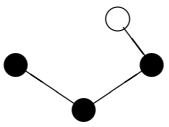
$$y_{n+1} = y_n + h\sum_{j=1}^{s} \alpha_j f(k_j) + \sum_{l=1}^{p}(\sum_{j=1}^{s} \gamma_j^{(l)} g(k_j))\theta_l \tag{5}$$

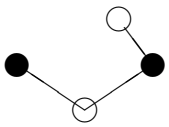
and its coefficients can be represented by the following tableau:

A	$B^{(1)}$	$\ldots$	$B^{(p)}$
α	$\gamma^{(1)}$	$\ldots$	$\gamma^{(p)}$

.

In this work we assume that $p \leq 2$ with $\theta_1 = J_{\{1\}}$ and $\theta_2 = \frac{J_{\{10\}}}{h}$. Pamela Burrage [4] developed a bi-colored rooted tree theory for the elementary differentials, elementary weights and order conditions for the derivation of stochastic RK methods by following the steps of deterministic RK Butcher tree theory [3, 6]. The bi-colored rooted trees T_b have two kind of vertices. Black vertices represent deterministic nodes and white vertices are used for stochastic nodes. Let $\emptyset$ the empty tree, τ the single deterministic vertex tree $\bullet$ and σ the single stochastic vertex tree $\circ$. These are enough to generate the set of bi-colored trees. We define the rest of the trees recursively so that $[t_1, t_2, \cdots, t_m] \in T_b$ when the root of the new tree is a deterministic node and $\{t_1, t_2, \cdots, t_m\} \in T_b$ then the root of the new tree is a stochastic node and $t_1, t_2, \ldots, t_m \in T_b$ e.g.:

$[\tau, [\sigma]]$ 

$\{\tau, [\sigma]\}$ 

The number of trees for the stochastic RK methods is considerably higher compared to the number of trees for the deterministic RKs. For each stochastic RK method we can define two kinds of orders. One is the deterministic order and the other is the strong stochastic order. There is a one to one correspondence between the elementary differentials, the trees and the order conditions. In such a tree we define its order ($ord(t)$) so that every deterministic node adds a unit and each stochastic one half. The deterministic order conditions for a stochastic RK method are the same as in the case of the pure deterministic RK methods [6]. For the general SDE we have to consider all the T_b trees (See Figure 1). Burrage's PhD thesis is an excellent source for the reader who wants to go further into the subject.

Hereafter, for simplification reasons, we introduce the following notation:

$$b = B^{(1)}e, c = Ae, d = B^{(2)}e. \tag{6}$$

For general problem (1) the following theorems hold.

Theorem 1 *Numerical method (5) where $p = 1$ and $\theta_1 = J_{\{1\}}$, has **stochastic order 1** when the following three order conditions hold:*

$$(\alpha^T e, \gamma^T e, \gamma^T b) = (1, 1, \frac{1}{2}). \tag{7}$$

We cannot have methods of higher order when $p = 1$, so in order to achieve stochastic order 1.5 we have to take $p = 2$.

Theorem 2 *Numerical method (5) where $p = 2$ with $\theta_1 = J_{\{1\}}$ and $\theta_2 = \frac{J_{\{10\}}}{h}$, has **stochastic order 1.5** when the following eighteen order conditions hold:*

$$\alpha^T(e, d, b) = (1, 1, 0)$$

$$\gamma^{(1)T}(e, d, b, c) = (1, -\gamma^{(2)T}b, \frac{1}{2}, 1)$$

$$\gamma^{(2)T}(e, d, c) = (0, 0, -1)$$

$$\gamma^{(1)T}(b^2, B^{(1)}b, d^2, B^{(2)}d) = (\frac{1}{3}, \frac{1}{6}, -2\gamma^{(2)T}bd, -\gamma^{(2)T}(B^{(2)}b + B^{(1)}d))$$

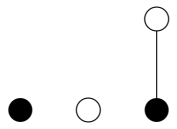
$$\gamma^{(2)T}(b^2, B^{(1)}b, d^2, B^{(2)}d) = (-2\gamma^{(1)T}bd, -\gamma^{(1)T}(B^{(2)}b + B^{(1)}d), 0, 0). \tag{8}$$

Tree		J	Φ	F
$z^{(0)^T} e \bullet a$	τ	$J(0)$	$z^{(0)^T} e$	a
$z^{(1)} e \circ b$	σ	$J(1)$	$z^{(1)^T} e$	b
$Z^{(0)} e \bullet a$ $2 z^{(1)^T} \circ b'$	$\{\tau\}$	$J(0,1)$	$2z^{(1)^T} Z^{(1)} e$	$b'(a)$
$Z^{(1)} e \circ b$ $2 z^{(0)^T} \bullet a'$	$[\sigma]$	$J(1,0)$	$2z^{(0)^T} Z^{(1)} e$	$a'(b)$
$Z^{(1)} e \circ b$ $2 z^{(1)^T} \circ b'$	$\{\sigma\}$	$J(1,1)$	$2z^{(1)^T} Z^{(1)} e$	$b'(b)$
$Z^{(1)} e \circ b \quad Z^{(1)} e \circ b$ $3 z^{(1)^T} \circ b''$	$\{\sigma, \sigma\}$	$J(1,1,1)$	$3z^{(1)^T} Z^{(1)} e * Z^{(1)} e$	$b''(b,b)$
$Z^{(1)} e \circ b$ $2Z^{(1)} e \circ b'$ $3 z^{(1)^T} \circ b'$	$\{\{\sigma\}\sigma\}$	$J(1,1,1)$	$6z^{(1)^T} Z^{(1)} e * Z^{(1)} e$	$b''(b(b))$

Figure 1: The stochastic integrals (J), the elementary weights () and the elementary differentials (F) for $t \in T_b : ord(t) \leq 1.5$

3 RK methods for SDEs with additive noise

The number of order conditions reduces significantly when we solve the SDE with additive noise. In such a case as many elementary differentials become zero and so, the corresponding order conditions do not play any role in the numerical process. In such a case, the only trees needed for a stochastic order 1.5 are the following:



$$t \in T_b : ord(t) \leq 1.5$$

and the following theorems hold.

Theorem 3 *Numerical method (5) where $p = 1$ and $\theta_1 = J_{\{1\}}$, when solving problem (3) has* ***stochastic order 1*** *when the following two order conditions hold:*

$$(\alpha^T e, \gamma^T e) = (1, 1). \tag{9}$$

Again, we cannot have methods of higher order when $p = 1$, so in order to achieve stochastic order 1.5 we have to take $p = 2$.

Theorem 4 *Numerical method (5) where $p = 2$ with $\theta_1 = J_{\{1\}}$ and $\theta_2 = \frac{J_{\{10\}}}{h}$, when solving problem (3) has **stochastic order 1.5** when the following five order conditions hold:*

$$(\alpha^T e, \gamma^{(1)T} e, \gamma^{(2)T} e, \alpha^T d, \alpha^T b) = (1, 1, 0, 1, 0). \tag{10}$$

3.1 Methods with two stages

For two stage methods we can easily achieve deterministic order two. Moreover we can satisfy the requirements of Theorem 3 and relation $\alpha c = \frac{1}{2}$. This equation minimizes the stochastic local truncation error coefficient that corresponds to the only tree with stochastic order 1.5 which is important for SDEs with additive noise i.e.

So, we can conclude to the following tree parameter family of methods:

$$\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ A_{21} & 0 & B_{21} & 0 \\ \hline 1-\frac{1}{2A_{21}} & \frac{1}{2A_{21}} & \gamma_1 & 1-\gamma_1 \end{array}.$$

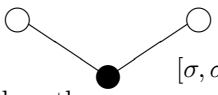
Such methods can be used to accompany higher order methods in embedded pairs.

3.2 Methods with three stages

In the families of three stage methods we can have deterministic order three. Moreover by using symbolic computations we can satisfy Theorem 4 order conditions resulting in a family of methods with ten free parameters.

$$\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & 0 & 0 & B_{21}^1 & 0 & 0 & B_{21}^2 & 0 & 0 \\ A_{31} & A_{32} & 0 & B_{31}^1 & B_{32}^1 & 0 & B_{31}^2 & B_{32}^2 & 0 \\ \hline \alpha_1 & \alpha_2 & \alpha_3 & \gamma_1^1 & \gamma_2^1 & \gamma_3^1 & \gamma_1^2 & \gamma_2^2 & \gamma_3^2 \end{array}.$$

Six of these parameters are involved in the deterministic truncation error coefficients and the relations $\alpha b^2 = \frac{1}{5}, \alpha d^2 = \frac{7}{5}, \alpha(b*d) = -\frac{1}{5}$. These equations minimize the stochastic local truncation error coefficient that corresponds to the only tree with stochastic order 2 which is important for SDEs with additive noise i.e.



If the above relations are satisfied, then the corresponding stochastic local truncation error coefficient minimizes to its minimum value 1/400. We have used these six parameters for the minimization of both deterministic and stochastic local truncation error coefficients and the remaining four parameters are free to be chosen and we choose them to satisfy:

$$\gamma^{(1)T}(b, c) = (\frac{1}{2}, 1)$$

$$\gamma^{(2)T}(d,c) = (0,-1).$$

We concluded to various methods but for our numerical tests we have chosen three of them. The first one we refer to as ADS3DO3SO15a, the second as ADS3DO3SO15b and the third as ADS3DO3SO15c. These attain norm of the deterministic principal local truncation error coefficient 0.043,0.04180 and 0.043 respectively. While the corresponding stochastic local truncation error coefficient approximate 1/400 with an error of 9×10^{-6}, 0.00170 and 10^{-44} respectively.

3.3 Methods with four stages

Analogously, by using symbolic computations, the families of methods with four stages can achieve deterministic order four and satisfy exactly Theorem 4 order conditions resulting in a family of methods with eighteen free parameters.

$$\begin{array}{cccc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{21} & 0 & 0 & 0 & B^1_{21} & 0 & 0 & 0 & B^2_{21} & 0 & 0 & 0 \\
A_{31} & A_{32} & 0 & 0 & B^1_{31} & B^1_{32} & 0 & 0 & B^2_{31} & B^2_{32} & 0 & 0 \\
A_{41} & A_{42} & A_{43} & 0 & B^1_{41} & B^1_{42} & B^1_{43} & 0 & B^2_{41} & B^2_{42} & B^2_{43} & 0 \\
\hline
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \gamma^1_1 & \gamma^1_2 & \gamma^1_3 & \gamma^1_4 & \gamma^2_1 & \gamma^2_2 & \gamma^2_3 & \gamma^2_4
\end{array}$$

Twelve of these parameters are involved in the deterministic truncation error coefficients and the relations $\alpha b^2 = \frac{1}{5}, \alpha d^2 = \frac{7}{5}, \alpha(b*d) = -\frac{1}{5}$. These equations minimize the stochastic local truncation error coefficient that corresponds to the only tree with stochastic order 2 which is important for SDEs with additive noise. If these relations are satisfied, then the corresponding stochastic local truncation error coefficient achieves its minimum value 1/400. We have used these twelve parameters for the minimization of both deterministic and stochastic local truncation error coefficients concluding to various methods. For our numerical tests we choose one four stage method which we refer to as ADS4DO4SO15. This method attains norm of the deterministic principal local truncation error coefficient 0.018 while the corresponding stochastic achieves 1/400 with an accuracy of forty six digits. As the remaining six parameters can be freely chosen we select them to satisfy:

$$\gamma^{(1)T}(b,c,b^2) = (\frac{1}{2}, 1, \frac{1}{3})$$

$$\gamma^{(2)T}(d,c,d^2) = (0,-1,0).$$

4 Numerical Tests

Using our methods ADS3DO3SO15a, ADS3DO3SO15b, ADS3DO3SO15c and ADS4DO4SO15 we have solved various SDE problems with additive noise found in the literature. We have compared their performance with two of Burrage's methods [4] called $E1$ and $E2$. These four stage methods are constructed for the solution of general SDEs and have stochastic order 1.5. As all the methods do not have the same number of stages we solve with different stepsizes so that in each trajectory all the methods perform the same number of function evaluations ($nofeptr$).

For our comparisons, as we do not know the analytical solution of the test problems, we consider as "analytical" solution the numerical solution using strong order 1.5 Taylor method with a very small stepsize ($h = 2^{-11}$ or $h = 2^{-9}$ in the third test problem). We perform M samples of N trajectories of the "analytical" X_T and numerical solution y^{δ}_T with common pseudo-random numbers in each trajectory. We measure the error $E(|X_T - y|)$ using the Monte Carlo method [2]. If we denote as $X_{T,k,j}$ and $y_{T,k,j}$ the k-trajectory values of the j-sample of the "analytical" and the numerical solution at time T and calculate

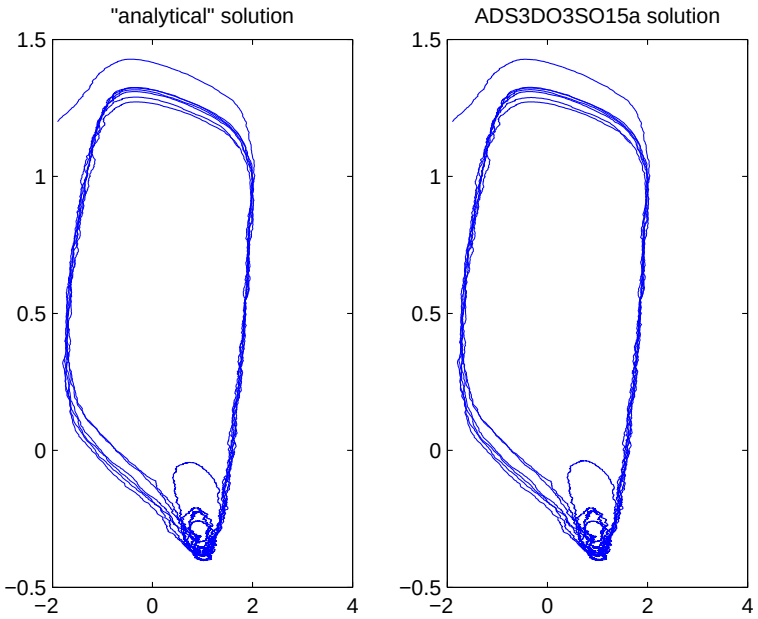


Figure 2: Bonhoeffer-Van der Pol Oscillator solution for $\sigma = 0.1$

$$\epsilon_j = \frac{1}{N}\sum_{k=1}^{N} |X_{T,k,j} - y_{T,k,j}|,$$

$$E(|X_T - y^\delta|) \simeq \epsilon = \frac{1}{M}\sum_{j=1}^{M} \epsilon_j,$$

$$\sigma_\epsilon^2 = \frac{1}{M-1}\sum_{j=1}^{M} (\epsilon_j - \epsilon)^2.$$

The trust region $100(1-a)\%$ of error ϵ has the form $(\epsilon - \Delta\epsilon, \epsilon + \Delta\epsilon)$ where $\Delta\epsilon = t_{1-a,M-1}\sqrt{\frac{\sigma_\epsilon^2}{M}}$. The quantity $t_{1-a,M-1}$ follows the t-Student distribution with $M-1$ degrees of freedom. More specifically, when $M = 20$ and $a = 0.1$ then $t_{1-a,M-1} \simeq 1.73$.

4.1 A stochastic logistic equation

The first problem is a model of adoption of products by Mahajan and Wind [7]:

$$dX_t^1 = c(R + L \cdot X_t) \cdot (K - X_t)dt + \sigma \circ dW_s$$

with $R = 0.0094, K = 18700, L = 0.3478/K, \sigma = 100, X_0 = 50$ when it refers to a sale of room air conditioners. The numerical results for $t \in [0, 3]$ are presented in Table 1.

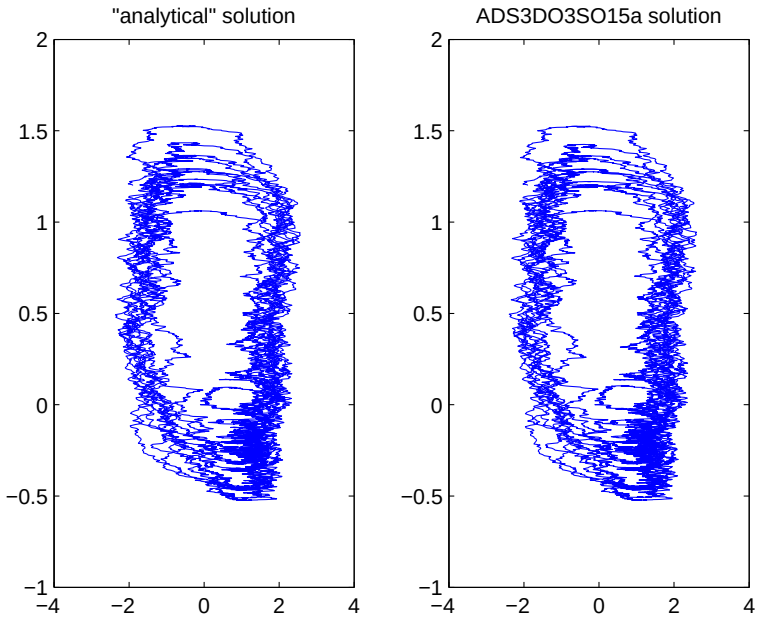


Figure 3: Bonhoeffer-Van der Pol Oscillator solution for $\sigma = 1$

Table 1: $E(|X_T - y|)$ for the stochastic logistic equation

$nofeptr$	1152	576	288	144
ADS3DO3SO15a	7.0611 ± 0.0577	7.1473 ± 0.1088	7.4392 ± 0.12942	9.8705 ± 0.2910
ADS3DO3SO15b	7.0613 ± 0.0577	7.1478 ± 0.1089	7.4418 ± 0.1294	9.8905 ± 0.2914
ADS3DO3SO15c	7.0611 ± 0.0577	7.1473 ± 0.1089	7.4401 ± 0.1662	9.8808 ± 0.2912
ADS4DO3SO15	7.0774 ± 0.0562	7.1915 ± 0.1519	8.0048 ± 0.1662	12.6637 ± 0.2427
E1	7.0721 ± 0.0559	7.1782 ± 0.1508	7.9421 ± 0.1622	12.3302 ± 0.2398
E2	7.0717 ± 0.0563	7.1822 ± 0.1521	8.0071 ± 0.1674	12.7511 ± 0.2429

4.2 An Experimental Psychology model

The second equation is a model by Schöner [1] where the stochastic part is included to describe fluctuations to the experimental data:

$$dX_t = -(asinX_t + 2bsin2X_t)dt + \sigma \circ dW_t$$

with $a = 1, b = 1, \sigma = 10, X_0 = 0$ and $t \in [0, 3]$. For this set of parameters the results of the numerical experiments are presented in Table 2.

4.3 Bonhoeffer-Van der Pol Oscillator

Finally, we solve a 2-dimensional simplification of a 4-dimensional ode system introduced by Hodgkin and Huxley [2] to model the firing of a single neuron. The effects of membrane imperfections and of the firing if nearby neurons can be simulated by the inclusion of the additive

Table 2: $E(|X_T - y|)$ for the Experimental Psychology model

$nofeptr$	1152	576	288	144
ADS3DO3SO15a	0.2181 ± 0.0071	0.3131 ± 0.0102	0.4385 ± 0.0135	0.6033 ± 0.0159
ADS3DO3SO15b	0.2208 ± 0.0069	0.3109 ± 0.0096	0.4348 ± 0.0147	0.5747 ± 0.0184
ADS3DO3SO15c	0.2142 ± 0.0070	0.3097 ± 0.0096	0.4327 ± 0.0118	0.6058 ± 0.0151
ADS4DO4SO15	0.3054 ± 0.0107	0.4567 ± 0.0117	0.6346 ± 0.0247	0.8637 ± 0.0308
E1	0.2930 ± 0.0077	0.4703 ± 0.0195	0.6626 ± 0.0196	0.8161 ± 0.0271
E2	0.2685 ± 0.0069	0.4197 ± 0.0154	0.6081 ± 0.0184	0.8333 ± 0.0202

noise term concluding to the following system:

$$dX_t^1 = c(X_t^1 + X_t^2 - \frac{1}{3}(X_t^1)^3 + z)dt + \sigma \circ dW_s$$

$$dX_t^2 = -\frac{1}{c}(X_t^1 + bX_t^2 - a)dt$$

with $a = 0.7, b = 0.8, c = 3.0, z = -0.34$, and $X_0 = [-1.9, 1.2]$.

We solve this demanding problem in the interval $t \in [0, 60]$, at a cost of 23049 $nofeptr$, for various values of the additive noise parameter σ and give the results in Table 3. Moreover two trajectories of the "analytical" solution and a solution with one of our methods is presented in Figures 2 and 3.

Table 3: $E(|X_T - y|)$ for various values of σ for neuron model

σ	0.1	0.5	1
ADS3DO3SO15a	0.1437 ± 0.0185	0.1016 ± 0.0263	0.0977 ± 0.0261
ADS3DO3SO15b	0.1436 ± 0.0186	0.1014 ± 0.0258	0.0993 ± 0.0266
ADS3DO3SO15c	0.1436 ± 0.0185	0.1026 ± 0.0268	0.0976 ± 0.0261
ADS4DO4SO15	0.1809 ± 0.0302	0.1110 ± 0.0283	0.1139 ± 0.0254
E1	0.1809 ± 0.0214	0.1127 ± 0.0283	0.1141 ± 0.0236
E2	0.1823 ± 0.0290	0.1314 ± 0.0268	0.1643 ± 0.0202

4.4 Remarks

As we have mentioned, our three stage methods, which are specially constructed for SDEs with additive noise, achieve stochastic order 1.5 just as the four stage methods do, at a lower cost per step. Our experiments showed that these methods outperformed all the four stage methods. Moreover, the loss in the deterministic order does not seem to affect their performance significantly.

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