European Society of Computational Methods in Sciences and Engineering (ESCMSE)

Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM)
vol. 15, no. 3-4, 2021, pp. 53-70
ISSN 1790–8140

A new twelfth order trigonometrically-fitted Obrechkoff-like method for second order initial value problems

Beny Neta *
Naval Postgraduate School
Department of Applied Mathematics
Monterey, CA 93943
e-mail: bneta@nps.edu, Tel: 1-831-656-2235, Fax: 1-831-656-2355

Received 01/02/2020, Revised 10/12/2020, Accepted 02/03/2021

Abstract: A new trigonometrically-fitted method of order 12 is developed and compared to an existing P-stable method of the same order. Our method fit exactly the sine and cosines functions $\sin(r\omega x)$, $\cos(r\omega x)$, $r = 1, 2$ and monomials up to degree 9. Our method is tested on several linear and non-linear examples to demonstrate its accuracy and sensitivity to perturbation in the known frequency. We also show where it is preferable to use the trigonometrically-fitted method. Our method shows its efficiency in solving a nonlinear equation both in terms of global accuracy and CPU run time.

© 2021 European Society of Computational Methods in Sciences and Engineering

*Corresponding author

1 Introduction

In this article we develop a new method for the numerical solution of the second order initial value problem

$$\begin{aligned} y'' &= f(x, y), \\ y(x_0) &= y_0, \\ y'(x_0) &= y'_0. \end{aligned} \tag{1}$$

Here we are concerned with trigonometrically-fitted methods for (1) whose solution is periodic or almost periodic with approximately known period.

There are several classes of methods, such as linear multistep methods (including Obrechkoff methods, see [16]) and Runge-Kutta methods. Another idea is the Adomian decomposition method [3] and its improvements [2]. See also [14] and [23] for methods applied to quantum chemistry. Here we develop a symmetric twelfth order scheme based on Obrechkoff methods to numerically solve problems for which the frequency is approximately known in advance, see review article [7].

Definition 1. Linear multistep methods for the solution (1) are given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k'} \beta_j f_{n+j} \tag{2}$$

where y_{n+j} is the approximate value at x_{n+j} and similarly for f_{n+j}. In here k is called the *step-number* and k' is either $k-1$ or k. In the former case the method is called *explicit* and in the latter it is called *implicit*. The coefficients α_j and β_j are chosen to satisfy stability and convergence, as we describe in the sequel.

We now introduce the *first* and *second characteristic polynomials*,

$$\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j \tag{3}$$

$$\sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j \tag{4}$$

Definition 2. The *order* of the linear multistep method (2) is defined to be p and its *error constant* to be C_{p+2} if, for an adequately smooth arbitrary test function $z(x)$

$$\sum_{j=0}^{k} [\alpha_j z(x+jh] - h^2\, \beta_j z''(x+jh)] = C_{p+2} h^{p+2} z^{(p+2)}(x) + O(h^{p+3}). \qquad (5)$$

The expression given by (5) is called the *local truncation error* at x_{n+k} of the method (2), when $z(x)$ is the theoretical solution of the initial value problem (1).

Throughout, we shall assume that the linear multistep method (2) satisfies the following hypotheses (see [12]):

- $\alpha_k = 1, |\alpha_0| + |\beta_0| \neq 0, \sum_{j=0}^{k} |\beta_j| \neq 0.$
- The characteristic polynomials ρ and σ have no common factors.
- $\rho(1) = \rho'(1) = 0, \rho''(1) = 2\sigma(1)$; this is necessary and sufficient for the method to be *consistent*, that is, to have order at least one.
- The method is *zero-stable*; that is, all the roots of ρ lie in or on the unit circle, those on the unit circle having multiplicity not greater than two.

We now consider the test equation (see e.g. Chawla and Neta [6])

$$y''(x) = -\lambda^2 y(x). \qquad (6)$$

Let $\zeta_s, s = 1, 2, \ldots, k$ denote the zeros of the polynomial

$$\Omega(\zeta, H^2) = \rho(\zeta) + H^2\sigma(\zeta) \qquad (7)$$

for $H = \lambda h$ and let ζ_1, ζ_2 correspond to perturbations of the principal roots of $\rho(\zeta)$. Then a linear multistep method is said to have an *interval of periodicity* $(0, H^2)$ if, for all H^2 in the interval, the roots ζ_s of (7) satisfy $\zeta_1 = e^{i\theta(H)}$, $\zeta_2 = e^{-i\theta(H)}$, $|\zeta_s| \leq 1,\ s \geq 3$ and $\theta(H)$ is real.

A linear multistep is called *P-stable* if its interval of periodicity is $(0, \infty)$. Lambert and Watson [13] had shown that a P-stable method is necessarily implicit of order at most 2.

Remark: The problem (1) has periodic solutions. If the period is not known, then the P-stability is desirable. If the period is known approximately, then one can use the ideas in Gautschi [9], Neta and Ford [15], and others.

Another important property when solving (1) is the *phase lag* which was introduced by Brusa and Nigro [5]. Upon applying a linear two-step method to the test equation (6), we obtain a difference equation of the form

$$A(H)y_{n+2} + B(H)y_{n+1} + C(H)y_n = 0 \tag{8}$$

whose solution is

$$y_n = B_1\lambda_1^n + B_2\lambda_2^n \tag{9}$$

where B_1 and B_2 are constants depending on the initial conditions. The quadratic polynomial

$$A(H)\lambda^2 + B(H)\lambda + C(H) = 0 \tag{10}$$

is called the *stability polynomial*. The solutions to (10) are given by

$$\begin{aligned} \lambda_1 &= e^{(-a(H)+ib(H))H} \\ \lambda_2 &= e^{(-a(H)-ib(H))H} \end{aligned} \tag{11}$$

If $a(H) \equiv 0$ and $b(H) \equiv 1$, then we get the exact solution to the test equation (6). The difference between the amplitudes of the exact solution of (6) and numerical solution is called *dissipation error*, see [10]. The frequency distortion depends on the magnitude $|b(H) - 1|$. The modulus of the leading term in the expansion of $b(H) - 1$ in powers of H is defined as the *phase lag* of the method and the expansion itself is called *phase lag expansion*. See also Thomas [19] and Twizell [20].

Simos [17] has developed a P-stable trigonometrically-fitted Obrechkoff method of algebraic order 10 for (1).

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{j=1}^{\ell} h^{2j}\left[b_{j\,0}\left(y_{n+1}^{(2j)} + y_{n-1}^{(2j)}\right) + 2b_{j\,1}y_n^{(2j)}\right], \tag{12}$$

where $\ell = 3$ and

$$\begin{aligned}
b_{1\,0} &= \frac{89}{1878} - \frac{15120}{313} b_{3\,1},\\
b_{1\,1} &= \frac{425}{939} + \frac{15120}{313} b_{3\,1},\\
b_{2\,0} &= -\frac{1907}{1577520} + \frac{660}{313} b_{3\,1},\\
b_{2\,1} &= \frac{30257}{1577520} + \frac{690}{313} b_{3\,1},\\
b_{3\,0} &= \frac{59}{3155040} - \frac{13}{313} b_{3\,1}.
\end{aligned} \tag{13}$$

In order to ensure P-stability, the coefficient $b_{3\,1}$ must be

$$\begin{aligned}
b_{3\,1} = \big(&190816819200[1-\cos(v)] - 95408409600v^2 + 7950700800v^4\\
&-265023360v^6 + 4732560v^8 - 52584v^{10} + 1727v^{12}\big) / (3568320v^{12}),
\end{aligned} \tag{14}$$

where $v = \omega h$. The method requires an approximation of the first derivative which is given by

$$y'_{n+1} = \frac{1}{2h}\left(y_{n-1} - 4y_n + 3y_{n+1}\right) - \frac{h}{12}\left(y''_{n-1} + 2y''_n - 3y''_{n+1}\right). \tag{15}$$

He showed that the local truncation error is

$$LTE = \left(-\frac{2923}{209898501120} + \frac{59}{1577520} b_{3\,1}\right) h^{12} y_n^{(12)}.$$

Wang et al. [22] have suggested a slight modification to the coefficient $b_{3\,1}$ as follows

$$b_{3\,1} = \frac{3155040 - 1428000v^2 + 60514v^4 - a_1\cos(v)}{5040v^2(-15120 + 6900v^2 - 313v^4 + a_2\cos(v))}, \tag{16}$$

where $a_1 = 3155040 + 149520v^2 + 3814v^4 + 59v^6$ and $a_2 = 15120 + 660v^2 + 13v^4$.

Wang et al. [22] have developed a method of algebraic order 12 as follows

$$\begin{aligned} y_{n+1} - 2y_n + y_{n-1} &= h^2 \left(\alpha_1 \left(y''_{n+1} + y''_{n-1}\right) + \alpha_2 y''_n\right) \\ &+ h^4 \left(\beta_1 \left(y^{(4)}_{n+1} + y^{(4)}_{n-1}\right) + \beta_2 y^{(4)}_n\right) \\ &+ h^6 \left(\gamma_1 \left(y^{(6)}_{n+1} + y^{(6)}_{n-1}\right) + \gamma_2 y^{(6)}_n\right), \end{aligned} \tag{17}$$

where

$$\alpha_1 = \frac{229}{7788}, \ \beta_1 = -\frac{1}{2360}, \ \beta_2 = \frac{711}{12980},$$

$$\gamma_1 = \frac{127}{39251520}, \ \gamma_2 = \frac{2923}{3925152},$$

and α_2 is chosen so the method is P-stable,

$$\alpha_2 = 2v^{-2} + v^2\beta_2 - v^4\gamma_2 + 2\cos(v)\left(-v^{-2} - \alpha_1 + v^2\beta_1 - v^4\gamma_1\right).$$

The method is of algebraic order 12 and the local truncation error is now

$$LTE = \frac{45469}{1697361329664000} h^{14}\left(\omega^{12} y''_n - y^{(14)}_n\right).$$

The first order derivative is obtained by

$$\begin{aligned} y'_{n+1} = &\frac{1}{66h}\left(305y_{n+1} - 544y_n + 239y_{n-1}\right) + \frac{h}{1980}\left(-5728y''_n - 571y''_{n-1}\right. \\ &\left.+119y''_{n+1}\right) + \frac{h^2}{2970}\left(128y'''_n - 173y'''_{n-1}\right) + \frac{h^3}{2970}\left(-346y^{(4)}_n - 13y^{(4)}_{n-1}\right) \\ &+\frac{h^5}{62370}\left(-71y^{(6)}_n + y^{(6)}_{n-1}\right). \end{aligned} \tag{18}$$

Remark: There were typographical errors in the coefficients given in [22] which were corrected in Chun and Neta [8].

2 New schemes

In this section, we develop a new symmetric Oberchkoff-type implicit scheme for second order systems of ordinary differential equations. The methods fit a combination of monomials and complex exponentials, i.e. the set

$$\{x^i\}_{i=0}^{K} \bigcup \{\sin(r\omega x), \cos(r\omega x)\}_{r=1}^{q}.$$

The new method is fitting monomials up to ninth degree ($K = 9$) and complex exponentials with $q = 2$.

The method is given by (12) with $\ell = 3$, i.e.

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{j=1}^{3} h^{2j} \left[b_{j\,0} \left(y_{n+1}^{(2j)} + y_{n-1}^{(2j)} \right) + 2b_{j\,1} y_n^{(2j)} \right]. \tag{19}$$

The set of equations to solve is Here we detail the development of our twelfth order method. We will list the resulting equations for the coefficients $b_{j\,0}$ and $b_{j\,1}$ for $j = 1, 2, 3$. Upon substituting the monomials x^0 and x^1, we find that the method (19) is satisfied automatically.

Substituting the monomial x^2 or x^3 leads to the same equation

$$-4b_{1\,0} - 4b_{1\,1} + 2 = 0 \tag{20}$$

Substituting $\sin(wx)$ or $\cos(wx)$ leads to

$$\begin{aligned} &(\cos(wh)b_{3\,0} + b_{3\,1})(wh)^6 - (\cos(wh)b_{2\,0} + b_{2\,1})(wh)^4 \\ &+(\cos(wh)b_{1\,0} + b_{1\,1})(wh)^2 + \cos(wh) - 1 = 0 \end{aligned} \tag{21}$$

Using the monomials x^4 leads to two equations, the first is identical to (20) and the other is

$$-24b_{1\,0} - 48b_{2\,0} - 48b_{2\,1} + 2 = 0 \tag{22}$$

The monomial x^5 give one new condition, namely

$$b_{3\,0} + b_{3\,1} = 0 \tag{23}$$

Now we substitute $\sin(2wx)$ and $\cos(2wx)$, the conditions are the same

$$
\begin{aligned}
&(256\cos(wh)^2 b_{3\,0} - 128 b_{3\,0} + 128 b_{3\,1})(wh)^6 \\
&+(-64\cos(wh)^2 b_{2\,0} + 32 b_{2\,0} - 32 b_{2\,1})(wh)^4 \\
&+(16\cos(wh)^2 b_{1\,0} - 8 b_{1\,0} + 8 b_{1\,1})(wh)^2 + 4\cos(wh)^2 - 4 = 0
\end{aligned} \tag{24}
$$

Using the monomial x^6, we have 3 conditions and only one of them is new, namely

$$
-60 b_{1\,0} - 720 b_{2\,0} - 1440 b_{3\,0} - 1440 b_{3\,1} + 2 = 0 \tag{25}
$$

The monomial x^7 yields multiples of the condition for x^6. The conditions resulting from the monomials x^8 and x^9 are equivalent and not new. Solving the equations (20) - (25) for the 6 coefficients, yields the result given in (26).

$$
\begin{aligned}
b_{1\,0} &= \frac{1}{60}\frac{(2c^2+40c+33)v^4 - (480c+465)v^2 - 945(c^2-1)}{(c^2+8c+6)v^4 + 15(c^2-1)v^2}, \\
b_{1\,1} &= \frac{1}{60}\frac{(28c^2+200c+147)v^4 + (450c^2+480c+15)v^2 + 945(c^2-1)}{(c^2+8c+6)v^4 + 15(c^2-1)v^2}, \\
b_{2\,0} &= \frac{1}{240}\frac{(-8c-7)v^4 + (10c^2+160c+145)v^2 + 315(c^2-1)}{(c^2+8c+6)v^4 + 15(c^2-1)v^2}, \\
b_{2\,1} &= \frac{1}{240}\frac{(6c^2+8c+1)v^4 + (140c^2+800c+635)v^2 + 1575(c^2-1)}{(c^2+8c+6)v^4 + 15(c^2-1)v^2}, \\
b_{3\,0} &= \frac{1}{240}\frac{(-2c^2+c+1)v^6 + (2c^3+28c^2+13c+47)v^4 + N_1}{(c-1)((c^2+8c+6)v^8 + (15c^2-15)v^6)}, \\
b_{3\,1} &= -b_{3\,0},
\end{aligned} \tag{26}
$$

where

$$
\begin{aligned}
&c = \cos(v), \\
&N_1 = (75c^3 + 15c^2 + 105c - 195)v^2 + 180(c^3 - c^2 - c + 1).
\end{aligned}
$$

The Taylor series expansion of the coefficients up to eighth order is

$$
\begin{aligned}
b_{1\,0} &= \tfrac{29}{600} + \tfrac{39}{88000}v^2 + \tfrac{7999}{411840000}v^4 + \tfrac{212777}{181209600000}v^6 + \tfrac{22554953}{287519232000000}v^8 + O\left(h^{10}\right), \\
b_{1\,1} &= \tfrac{271}{600} - \tfrac{39}{88000}v^2 - \tfrac{7999}{411840000}v^4 - \tfrac{212777}{181209600000}v^6 - \tfrac{22554953}{287519232000000}v^8 + O\left(h^{10}\right), \\
b_{2\,0} &= -\tfrac{1}{800} - \tfrac{13}{352000}v^2 - \tfrac{7999}{4942080000}v^4 - \tfrac{212777}{2174515200000}v^6 - \tfrac{22554953}{3450230784000000}v^8 + O\left(h^{10}\right), \\
b_{2\,1} &= \tfrac{3}{160} - \tfrac{13}{70400}v^2 - \tfrac{7999}{988416000}v^4 - \tfrac{212777}{434903040000}v^6 - \tfrac{22554953}{690046156800000}v^8 + O\left(h^{10}\right), \\
b_{3\,0} &= \tfrac{59}{3024000} + \tfrac{13}{7040000}v^2 + \tfrac{96553}{691891200000}v^4 + \tfrac{569914291}{57537672192000000}v^6 + \tfrac{1280357251}{1863124623360000000}v^8 \\
&\quad + O\left(h^{10}\right), \\
b_{3\,1} &= -b_{3\,0}.
\end{aligned} \tag{27}
$$

We now apply our method (19) to test equation (6), see e.g. [1]. We obtain the following difference equation

$$A(v)\xi^2 - 2B(v)\xi + A(v) = 0, \tag{28}$$

where

$$
\begin{aligned}
A &= 1 + b_{1\,0}v^2 - b_{2\,0}v^4 + b_{3\,0}v^6, \\
B &= 1 - b_{1\,1}v^2 + b_{2\,1}v^4 - b_{3\,0}v^6.
\end{aligned} \tag{29}
$$

The phase lag is given by

$$\frac{7993450002076379}{4945603202221473792000000000}v^{18}.$$

For the local truncation error see [11].

3 Numerical examples

In this section we compare our new scheme for second order systems, denoted OM3, to the method due to Wang et al., denoted Wang, for the solution of several examples. For both methods we use (18) to approximate the first derivative.

Example 1

In our first example, we use the almost periodic problem

$$y''(x) + \left(100 + \frac{1}{4x^2}\right) y(x) = 0, \ 1 \leq x \leq 100. \tag{30}$$

We choose the initial condition so that

$$y_{exact}(x) = \sqrt{x} J_0(10x),$$

and pick $\omega = 10$.

The results are given in Table 1 at $x = 100$. Both methods gave similar results for the larger h, i.e. of order 10^{-18}. The error is of order of 10^{-18} for our method and 10^{-17} for Wang. The CPU time to run the code is slightly lower using our method.

Method	L_2 Error for $h = 0.02$	L_2 Error for $h = 0.002$	CPU for $h = 0.002$
OM3	0.333105(-10)	0.337424(-18)	18.771
Wang	0.877418(-10)	0.110750(-17)	18.795

Table 1: The L_2 error and CPU of both methods at $x = 100$ where the exact solution is 0.0798900501

Example 2

A second example is

$$y^{(4)} + 2y'' + y = \sin(x). \tag{31}$$

Initial conditions

$$y(0) = 1, \ y'(0) = 1, \ y''(0) = 1, \ y'''(0) = 1. \tag{32}$$

$$y_{exact} = \cos(x) + \frac{19}{8}\sin(x) + x\left(\sin(x) - \frac{11}{8}\cos(x)\right) - \frac{1}{8}x^2\sin(x). \tag{33}$$

We can rewrite this as a system of 2 second order initial value problems

$$\begin{aligned} y_1'' &= y_2, \\ y_2'' &= -2y_2 - y1 + \sin(x), \end{aligned} \tag{34}$$

$$y_1(0) = 1,\ y_2(0) = 1. \tag{35}$$

The results at $x = 40\pi$ using three values of h are given in Table 2. Again the error is of order 10^{-18} for both methods. The error for our method is slightly smaller. Our method is again slightly faster.

Method	$h = \frac{\pi}{50}$	$h = \frac{\pi}{250}$	$h = \frac{\pi}{500}$	CPU for $h = \frac{\pi}{500}$
OM3	0.999827(-11)	0.102514(-14)	0.641287(-15)	10.755
Wang	0.265514(-10)	0.266814(-15)	0.889702(-15)	10.902

Table 2: The L_2 error of both methods at $x = 40\pi$ using three values of time steps

Example 3

In our third example we took

$$\begin{aligned} &z''(x) + z(x) = 0.001e^{ix}, \\ &z(0) = 1, \\ &z'(0) = 0.9995i. \end{aligned} \tag{36}$$

This example should give a leg up to methods using $x\sin(\omega x)$ and $x\cos(\omega x)$, as can be seen in the exact solution below.

The exact solution is

$$\begin{aligned} &z(x) = u(x) + iv(x), \\ &u(x) = \cos(x) + 0.0005x\sin(x), \\ &v(x) = \sin(x) - 0.0005x\cos(x). \end{aligned} \tag{37}$$

We solved the equation on the interval $0 \le x \le 40\pi$ using $\omega = 1$ and compared the exact value of γ defined as $\gamma = \sqrt{u^2 + v^2} = \sqrt{1 + (0.0005x)^2}$ to the approximate value at the end of the interval using two values of $h = \pi/25, \pi/50$. The results for both methods are given in Table 3. Again the method gave similar result, but the accuracy now is lower

Method	$h = \frac{\pi}{25}$	$h = \frac{\pi}{50}$
OM3	0.215540(-11)	0.839028(-14)
Wang	0.569522(-11)	0.222028(-13)

Table 3: The L_2 error of both methods at $x = 40\pi$ using two values of time steps

Example 4

In our fourth example, we consider the nonlinear undamped Duffing's equation, see e.g. van Dooren [18] and Vanden Berghe and Van Daele [4]

$$y'' + y + y^3 \;=\; B\cos(\Omega t),$$

with $B = .002$ and $\Omega = 1.01$. The exact solution (see Vanden Berghe and Van Daele [4]) is given by

$$y(t) = A_1\cos(\Omega t) + A_3\cos(3\Omega t) + A_5\cos(5\Omega t) + A_7\cos(7\Omega t) + A_9\cos(9\Omega t),$$

where

$$\begin{aligned} A_1 &= 0.2001794775361502, \\ A_3 &= 2.46946143255559(-4), \\ A_5 &= 3.0401498519692437(-7), \\ A_7 &= 3.743490701609247(-10), \\ A_9 &= 4.609682949622697(-13). \end{aligned}$$

The initial conditions are

$$y(0) = A_1 + A_3 + A_5 + A_7 + A_9,$$

$$y'(0) = 0.$$

The results are given in Table 4. Clearly the methods are comparable.

Example 5

Method	$h = \frac{\pi}{50}$
OM3	0.134043(-11)
Wang	0.134252(-11)

Table 4: The L_2 error of both methods at $x = 40\pi$.

In our fifth example, we consider the Mathieu differential equation

$$y''(x) + 100\,(1 - \alpha\cos(2x))\,y(x) = 0,$$

with boundary conditions

$$y(0) = 1,$$
$$y'(0) = 0,$$

and $\alpha = 0.1$. The frequency is $\pi/5$ (see Gautschi [9]). Mathieu equation appears in physical problems involving elliptical shapes or periodic potentials. We will solve the equation for $0 \le x \le 5$ and tabulate at every unit. The results are compared to the values obtained from Maple using the function MathieuC(100,5,x). The results are identical for both methods. The L_2 error is computed at the end of integration interval, i.e. $x = 5$.

Method	$x = 1.0$	$x = 25.0$	$x = 50.0$	$x = 75.0$	$x = 100.0$	L_2 error
OM3	-0.908418	0.151668	-0.951685	-0.4205942	0.8338947	0.167908(-8)
Wang	-0.908418	0.151668	-0.951685	-0.420594	0.833894	0.101908(-8)
Exact	-0.908418	0.151668	-0.951685	-0.420594	0.833894	

Table 5: The approximate solution using both methods (with $h = 0.02$) at various values of the independent variable. The exact value at each point was found using the Maple function MathieuC.

Example 6 The last example is the cubic oscillator as given in [21]

$$y''(x) + y(x) = \epsilon y(x)^3, \qquad \epsilon = 10^{-3}, \tag{38}$$

with the initial conditions

$$\begin{aligned} y(0) &= 1, \\ y'(0) &= 0, \end{aligned} \tag{39}$$

and the frequency $\omega = \sqrt{1 - 0.75\epsilon}$. The exact solution to cubic order in ϵ is given in [21]

$$y(x) = \cos(\omega x) + \frac{\epsilon}{128}\left(\cos(3\omega x) + \cos(\omega x)\right) + O\left(\epsilon^3\right).$$

The results are given in Table 6. It is clear that both methods gave identical results. Since the exact solution is only accurate to order 10^{-9}, the L_2 norm is higher than in previous examples. The error is computed at $x = 2000\pi$. This also demonstrate the capability of both method to due a long term integration.

Method	L_2 Error	CPU time
OM3	0.555667(-7)	112.967
Wang	0.555667(-7)	119.634

Table 6: The L_2 norm of the error and CPU for the sixth example.

Based on these results, it seems that the trigonometrically-fitted method we developed is as good as the P-stable method of Wang et al. But it is faster than Wang's method by 5.9%.

We now consider two other examples to see if this is always the case. The first one is to check the sensitivity to a small perturbation in the frequency ω. This is important when the frequency is not known exactly. In example 1, the solution is not periodic but almost periodic. Nevertheless, the solution is very accurate. We re-ran Examples 1, 4 and 6 using $\omega \pm 0.1$ and the results are summarized in Table 7.

From Table 7 it is clear that Wang is not sensitive to a small perturbation in ω. Our method, OM3, for example 1 seems slightly more sensitive. Notice that for example 6 both method lost 2 significant figures by perturbing ω. Note that perturbing ω by 0.1 is 10% in examples 4 and 6 but it is only 1% in example 1. In all the examples we see that underestimating or overestimating does NOT affect the accuracy.

Example	Method	ω	L_2 Error
1	OM3	9.9	0.145272(-10)
1	OM3	10.	0.136230(-10)
1	OM3	10.1	0.127096(-10)
1	Wang	9.9	0.358570(-10)
1	Wang	10	0.358570(-10)
1	Wang	10.1	0.358570(-10)
4	OM3	.9	0.315810(-13)
4	OM3	1.	0.315988(-13)
4	OM3	1.1	0.316148(-13)
4	Wang	0.9	0.315599(-13)
4	Wang	1	0.315599(-13)
4	Wang	1.1	0.315599(-13)
6	OM3	$\omega - 0.1$	0.207324(-5)
6	OM3	ω	0.5556678(-7)
6	OM3	$\omega + 0.1$	0.207324(-5)
6	Wang	$\omega - 0.1$	0.2040854(-5)
6	Wang	ω	0.5556678(-7)
6	Wang	$\omega + 0.1$	0.2040854(-5)

Table 7: The sensitivity to small perturbations in ω.

Now we add another example taken from Chun and Neta [7].

Example 7

This example is chosen so that the exact solution is a combination of sine and cosine of multiples of the frequency, i.e.

$$y''(x) + 9y(x) = 3\sin(6x),\ \ 0 \le x \le 40\pi, \tag{40}$$

subject to the initial conditions

$$y(0) = 1,\ \ y'(0) = 3. \tag{41}$$

The exact solution is

$$y_{exact}(x) = \frac{11}{9}\sin(3x) + \cos(3x) - \frac{1}{9}\sin(6x). \tag{42}$$

The results using $h = \pi/500$ are given in Table 8.

Method	ω	L_2 error	Global Error
OM3	3.	0.205241(-17)	0.144838(-17)
Wang	3	0.172924(-12)	0.958545(-7)

Table 8: The L_2 norm of the error and global error for example 7.

This example demonstrates the superiority of our trigonometrically-fitted method over a P-stable method of the same order using the same frequency. Notice that the global error is **much** smaller by 10 orders of magnitude.

Conclusions We have developed a twelfth order trigonometrically-fitted method for approximating the solution of a second order initial value problems whose solution is periodic with a known period. We have compared our method to a twelfth order P-stable method. We have shown that the accuracy is the same except in one case. If the exact solution is a sum of trigonemtric functions of different period, then our method is more accurate. The CPU run time required by our method to run example 6 (nonlinear equation) is much less than Wang's. The last example demonstrates the superiority of our method in terms of global accuracy.

References

[1] U. Ananthakrishnaiah, P-stable Obrechkoff methods with minimal phase-lag for periodic initial value problems, Math. Comp., 49, (1987), 553—559.

[2] T. A. Abassy, Improved Adomian decomposition method, Comput. Math. Applic., 59, (2010), 42—54.

[3] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, vol. 60 of Fundamental Theories of Physics, Kluwer Academic, Dordrecht, The Netherlands, 1994.

[4] G. Vanden Berghe, M. Van Daele, Exponentially-fitted Obrechkoff methods for second-order differential equations, Appl. Numer. Math., 59, (2009), 815–829.

[5] L. Brusa, L. Nigro, A one-step method for the direct integration of structural dynamic equations, Internat. J. Numer. Meth. Engng., 15, (1980), 685-699.

[6] M. M. Chawla, B. Neta, Families of two-step fourth order P-stable methods for second order differential equations, J. Comput. Appl. Math., 15, (1986), 213-223.

[7] C. Chun, B. Neta, Trigonometrically-fitted method: a review, MDPI Mathematics, Mathematics, 7(12), (2019), 1197—1216.

[8] C. Chun, B. Neta, A new trigonometrically-fitted method for second order initial value problems, Naval Postgraduate School Technical Report, NPS-MA-20-001, Monterey, CA.

[9] W. Gautschi, Numerical integration of ordinary differential equations based on trigonometric polynomials, Numer. Math., 3, (1961), 381-397.

[10] P. J. Van der Houwen, B. P. Sommeijer, Explicit Runge-Kutta-Nyström methods with reduced phase errors for computing oscillating solutions, SIAM J. Numer. Anal., 24, (1987), 595-617.

[11] I. Gr. Ixaru, G. vanden Berghe, Exponential fitting, Mathematics and Its Application, vol. 568, Kluwer Academic Publications, 2004.

[12] J. D. Lambert, *Numerical methods for ordinary differential systems: the initial value problem*, John Wiley & Sons Ltd., England, 1991.

[13] J. D. Lambert, I. A. Watson, Symmetric multistep methods for periodic initial value problems, J. Inst. Math. Appl., 18, (1976), 189–202

[14] M. A. Medvedev, T. E. Simos, New FD scheme with vanished phase-lag and its derivatives up to order six for problems in chemistry, J. Math. Chem., 58, (2020), 2324–2360.

[15] B. Neta, C. H. Ford, Families of methods for ordinary differential equations based on trigonometric polynomials, J. Comput. Appl. Math., 10, (1984), 33–38.

[16] N. Obrechkoff, On mechanical quadrature (Bulgarian, French summary), Spisanie Bulgar. Akad. Nauk, 65, (1942), 191–289.

[17] T. E. Simos, A P-stable complete in phase Obrechkoff trigonometric fitted method for periodic initial-value problems, Proc. R. Soc. Lond. A, 441, (1993), 283–289.

[18] R. Van Dooren, Stabilization of Cowell's classical finite difference method for numerical integration, J. Comput. Phys., 16, (1974), 186–192.

[19] R. M. Thomas, Phase properties of high order, almost P-stable formulae, BIT, 24, (1984), 225-238.

[20] E. H. Twizell, Phase-lag analysis for a family of two-step methods for second order periodic initial value problems, J. Comput. Appl. Math., 15, (1986), 261-263.

[21] J. Vigo-Aguiar, T. E. Simos, J. M. Ferrándiz, Controlling the error growth in long-term numerical integration of perturbed oscillations in one or several frequencies, Proc. Math. Phys. Eng. Sci., 460, (2004), 561–567.

[22] Z. Wang, D. Zhao, Y. Dai, D. Wu, An improved trigonometrically fitted P-stable Obrechkoff method for periodic initial-value problems, Proc. Math. Phys. Engng. Sci., 461, (2005), 1639–1658.

[23] Z. Wang, T. E. Simos, A new algorithm with eliminated phase-lag and its derivatives up to order five for problems in quantum chemistry, J. Math. Chem., 58, (2020), 2361–2398.