European Society of Computational Methods in Sciences and Engineering (ESCMSE)

Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM)
vol. 12, no. 3-4, 2018, pp. 1-19
ISSN 1790–8140

Modified Finite Integration Method Using Chebyshev Polynomial for Solving Linear Differential Equations

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Received 10 January 2018; accepted in revised form 27 August 2018

Abstract: We propose a modified finite integration method (FIM) by using the Chebyshev polynomial, to construct the first order integral matrix for solving linear differential equations in one and two dimensions. The grid points for the computation are generated by the zeros of the Chebyshev polynomial of a certain degree. We implement our method with several examples arose from real-world applications. In comparison with the finite difference method and the traditional FIMs (trapezoidal and Simpson's rules), numerical computations show that our modified FIM using Chebyshev nodes require the less computational cost to achieve significant improvement in accuracy.

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Keywords: Finite integration method, Linear differential equations, Chebyshev polynomial.

Mathematics Subject Classification: 65L05, 65L10, 65N30

1 Introduction

Most natural and physical phenomena such as the moving storm, the movement of lithosphere, wave, heat, sound and electricity, etc. are well described by differential equations. When the analytical solution of a mathematically defined problem is impossible or possible, but with time-consuming, the numerical methods play an important role in finding *good* approximate solutions to the problem. There are many numerical methods available for solving differential equations such as finite difference method (FDM), finite element method (FEM) and boundary element method (BEM), etc., see [1].

Recently, the finite integration method (FIM) has been developed for finding approximate solutions to linear boundary value problems for ordinary and partial differential equations. In 2013, Wen et al. [5] developed the FIM with ordinary linear approximation and radial basis function. In their work, they used the integration matrix of the first order to obtain the integration matrix with

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multi-layers. In 2015, Li et al. [3] extended the FIM with ordinary linear approximation and radial basis function to solve multi-dimensional differential equations. In 2016, Li et al. [4] improved the FIM using three numerical quadrature formulas which are Simpson's rule, Cotes integral, and Lagrange interpolation. They showed that the improved FIM with Simpson's rule and Lagrange interpolation were highly accurate compared with the FDM.

In this paper, we modify and improve the FIM by using the orthogonal function, which is the Chebyshev polynomial. By using the Chebyshev nodes to interpolate the approximate solution, hiring some properties of Chebyshev polynomials, and modifying the ideas of Li et al. [3] and Wen et al. [5], we can construct the first and higher order integral matrices for solving linear differential equations in one and two dimensional problems. We note here on the major difference of our method and those of Li et al. [3] and Wen et al. [5] that, by using the zeros of the Chebyshev polynomials to generate grid points, the domain is discretized based on these nodes which gives unequal discretization while [3] and [5] use equal discretization. Finally, we implement our modified FIM with several numerical examples to demonstrate the accuracy of our modified FIM compare with the FDM and the traditional FIMs proposed by Wen et al. [5] and Li et al. [4] and their analytical solutions.

2 Modified FIM by using Chebyshev polynomials

For ease of reference, we provide the definition and some important properties of the Chebyshev polynomial $T_n(x)$. The properties in Lemma 2.2 will be used to construct the first and higher order integral matrices for solving ordinary and partial differential equations (ODEs and PDEs).

Definition 2.1 ([2]) *The Chebyshev polynomial of degree $n \geq 0$ is defined as*

$$T_n(x) = \cos(n \arccos x) \text{ for } x \in [-1, 1]. \tag{1}$$

Lemma 2.2 (i) *For $n \in \mathbb{N}$, the zeros of Chebyshev polynomial $T_n(x)$ are*

$$x_k = \cos\left[\frac{(2k-1)\pi}{2n}\right], \; k \in \{1, 2, 3, ..., n\}. \tag{2}$$

(ii) *For $p \in \mathbb{N}$, the p^{th} order derivatives of Chebyshev $T_n(x)$ at $x = \pm 1$ are*

$$\frac{d^p}{dx^p} T_n(x)\Big|_{x=\pm 1} = (\pm 1)^{p+n} \prod_{k=0}^{p-1} \frac{n^2 - k^2}{2k+1}. \tag{3}$$

(iii) *For $n \in \mathbb{N}$, the discrete orthogonality relation of $T_i(x_k)$ and $T_j(x_k)$ is*

$$\sum_{k=1}^{n} T_i(x_k) T_j(x_k) = \begin{cases} 0 & \text{if } i \neq j, \\ n & \text{if } i = j = 0, \\ n/2 & \text{if } i = j \neq 0, \end{cases}$$

where $x_k, k \in \{1, 2, 3, ..., n\}$, satisfies (2), and $0 \leq i, j \leq n$.
(iv) *For $n \in \mathbb{N}$ and $x \in [-1, 1]$, the single layer integrations of $T_n(x)$ are*

$$\bar{T}_0(x) = \int_{-1}^{x} T_0(\xi)\, d\xi = x + 1,$$

$$\bar{T}_1(x) = \int_{-1}^{x} T_1(\xi)\, d\xi = \frac{x^2}{2} - \frac{1}{2},$$

$$\bar{T}_n(x) = \int_{-1}^{x} T_n(\xi)\, d\xi = \frac{1}{2}\left[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right] - \frac{(-1)^n}{n^2-1} \text{ for } n \geq 2.$$

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(v) *Let* $\{x_k\}_{k=1}^n$ *be the zeros of* $T_n(x)$ *and define the Chebyshev matrix* $\boldsymbol{T}$ *by*

$$\boldsymbol{T} = \begin{bmatrix} T_0(x_1) & T_1(x_1) & \cdots & T_{n-1}(x_1) \\ T_0(x_2) & T_1(x_2) & \cdots & T_{n-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_n) & T_1(x_n) & \cdots & T_{n-1}(x_n) \end{bmatrix}.$$

Then, it has the multiplicative inverse $\boldsymbol{T}^{-1} = \frac{1}{n}\mathrm{diag}(1,2,2,...,2)\,\boldsymbol{T}^T$.

2.1 Modified FIM in one dimension

Let N be a positive integer and $u(x)$ be a linear combination of the Chebyshev polynomials $T_0(x), T_1(x), T_2(x), ..., T_{N-1}(x)$, that is,

$$u(x) = \sum_{n=0}^{N-1} c_n T_n(x) \text{ for } x \in [-1, 1]. \tag{4}$$

Let $-1 \le \bar{x}_1 < \bar{x}_2 < \bar{x}_3 < ... < \bar{x}_N \le 1$ be grid points generated by the zeros of Chebyshev polynomial $T_N(x)$ defined in (2). By (4), we have

$$\begin{bmatrix} u(\bar{x}_1) \\ u(\bar{x}_2) \\ \vdots \\ u(\bar{x}_N) \end{bmatrix} = \begin{bmatrix} T_0(\bar{x}_1) & T_1(\bar{x}_1) & \cdots & T_{N-1}(\bar{x}_1) \\ T_0(\bar{x}_2) & T_1(\bar{x}_2) & \cdots & T_{N-1}(\bar{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(\bar{x}_N) & T_1(\bar{x}_N) & \cdots & T_{N-1}(\bar{x}_N) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix},$$

which is denoted by $\mathbf{u} = \mathbf{Tc}$. Thus, the coefficients $\{c_n\}_{n=0}^{N-1}$ can be determined by $\mathbf{c} = \mathbf{T}^{-1}\mathbf{u}$. For $k \in \{1, 2, 3, ..., N\}$, let us consider the single layer integration of $u(x)$ from -1 to $\bar{x}_k$, which is denoted by $U(\bar{x}_k)$. Then, we have

$$U(\bar{x}_k) = \int_{-1}^{\bar{x}_k} u(\xi)\, d\xi = \sum_{n=0}^{N-1} c_n \int_{-1}^{\bar{x}_k} T_n(\xi)\, d\xi = \sum_{n=0}^{N-1} c_n \bar{T}_n(\bar{x}_k),$$

for $k \in \{1, 2, 3, ..., N\}$ or

$$\begin{bmatrix} U(\bar{x}_1) \\ U(\bar{x}_2) \\ \vdots \\ U(\bar{x}_N) \end{bmatrix} = \begin{bmatrix} \bar{T}_0(\bar{x}_1) & \bar{T}_1(\bar{x}_1) & \cdots & \bar{T}_{N-1}(\bar{x}_1) \\ \bar{T}_0(\bar{x}_2) & \bar{T}_1(\bar{x}_2) & \cdots & \bar{T}_{N-1}(\bar{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{T}_0(\bar{x}_N) & \bar{T}_1(\bar{x}_N) & \cdots & \bar{T}_{N-1}(\bar{x}_N) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix}.$$

We will denote the above equation by $\mathbf{U} = \bar{\mathbf{T}}\mathbf{c} = \bar{\mathbf{T}}\mathbf{T}^{-1}\mathbf{u} = \mathbf{Au}$, where we have let $\mathbf{A} = \bar{\mathbf{T}}\mathbf{T}^{-1}$. The matrix $\mathbf{A} = \bar{\mathbf{T}}\mathbf{T}^{-1} := [a_{ki}]_{N\times N}$ is called *the first order integral matrix* for our modified FIM in one dimension.

By using the idea of Wen et al. [5] and Li et al. [4], we can calculate the m-layer integration,

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$U^{(m)}(\bar{x}_k)$, of $u(x)$ from -1 to $\bar{x}_k$ by

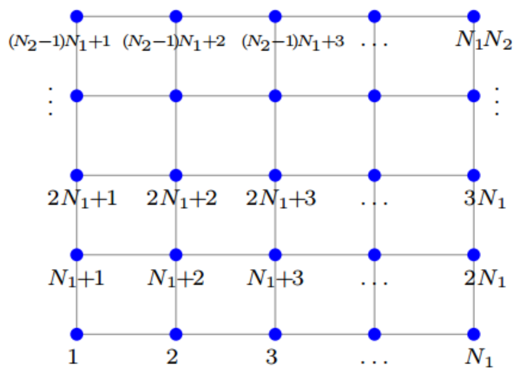
$$\begin{aligned} U^{(m)}(\bar{x}_k) &= \int_{-1}^{\bar{x}_k} \cdots \int_{-1}^{\xi_2} u(\xi_1)\, d\xi_1 \ldots d\xi_m \\ &= \sum_{i_m=1}^{N} \cdots \sum_{j=1}^{N} a_{ki_m} \ldots a_{i_1 j} u(\bar{x}_j) \\ &= \sum_{i=1}^{N} a_{ki}^{(m)} u(\bar{x}_i), \end{aligned}$$

whose matrix form can be expressed as $\mathbf{U}^{(m)} = \mathbf{A}^{(m)}\mathbf{u} = \mathbf{A}^m\mathbf{u}$, $a_{ki}^{(m)}$ is the i^{th} element in row k of A^m and $\mathbf{U}^{(m)} = [U^{(m)}(\bar{x}_1), U^{(m)}(\bar{x}_2), U^{(m)}(\bar{x}_3), ..., U^{(m)}(\bar{x}_N)]^T$. The matrix $\mathbf{A}^{(m)}$ is called *the m^{th} order integral matrix* for our modified FIM in one dimension.

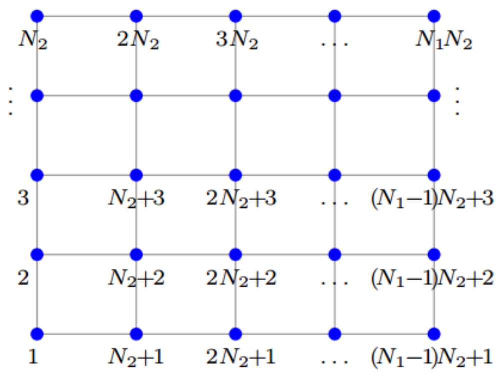
2.2 Modified FIM in two dimension

Let $\Omega = [a, b] \times [c, d]$, where $a < b$ and $c < d$ are real. To use our proposed method over Ω, we first transform Ω into $\bar{\Omega} = [-1, 1] \times [-1, 1]$. The Chebyshev nodes defined in (2) will be used to discretize with $N_1 \times N_2$-point discretization, where N_1 and N_2 are the total number of horizontal and vertical discretization points, respectively.

For computational convenience, we index the numbering of grid points along the x-direction by the global numbering system (Figure 1a) and grid points along the y-direction by the local numbering system (Figure 1b).



(a) Global numbering system



(b) Local numbering system

Figure 1: The indices of the grid points globally and locally

Let $-1 \le \bar{x}_1 < \bar{x}_2 < ... < \bar{x}_{N_1} \le 1$ and $-1 \le \bar{y}_1 < \bar{y}_2 < ... < \bar{y}_{N_2} \le 1$ be grid points along the x- and y-directions, respectively, that are generated by the zeros of Chebyshev polynomials T_{N_1} and T_{N_2} as defined in (2).

Let $U_x(x, y)$ and $U_y(x, y)$ be the single layer integrations with respect to the variables x and y, respectively. Then, for each fixed $\bar{y}_s$, we have $U_x(\bar{x}_k, \bar{y}_s)$ in the global numbering system as

$$U_x(\bar{x}_k, \bar{y}_s) = \int_{-1}^{\bar{x}_k} u(\xi, \bar{y}_s)\, d\xi = \sum_{i=1}^{N_1} a_{ki} u(\bar{x}_i, \bar{y}_s), \tag{5}$$

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for $k \in \{1, 2, 3, ..., N_1\}$ or $\mathbf{U}_x(\cdot, \bar{y}_s) = \mathbf{A}\mathbf{u}(\cdot, \bar{y}_s)$, where $\mathbf{A} = \bar{\mathbf{T}}\mathbf{T}^{-1}$ with size $N_1 \times N_1$. Thus, for $s \in \{1, 2, 3, ..., N_2\}$,

$$\begin{bmatrix} \mathbf{U}_x(\cdot, \bar{y}_1) \\ \mathbf{U}_x(\cdot, \bar{y}_2) \\ \vdots \\ \mathbf{U}_x(\cdot, \bar{y}_{N_2}) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & 0 & \cdots & 0 \\ 0 & \mathbf{A} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{A} \end{bmatrix}}_{N_2 \text{ blocks}} \begin{bmatrix} \mathbf{u}(\cdot, \bar{y}_1) \\ \mathbf{u}(\cdot, \bar{y}_2) \\ \vdots \\ \mathbf{u}(\cdot, \bar{y}_{N_2}) \end{bmatrix}, \tag{6}$$

or $\mathbf{U}_x = \mathbf{A}_x\mathbf{u}$. The matrix $\mathbf{A}_x$ is called *the first order integral matrix with respect to x* for our midified FIM in two dimension.

Similarly, for each fixed $\bar{x}_k$, $U_y(\bar{x}_k, \bar{y}_s)$ can be expressed in the local numbering system as

$$U_y(\bar{x}_k, \bar{y}_s) = \int_{-1}^{\bar{y}_s} u(\bar{x}_k, \eta)\, d\eta = \sum_{j=1}^{N_2} a_{sj} u(\bar{x}_k, \bar{y}_j),$$

for $s \in \{1, 2, 3, ..., N_2\}$ or $\tilde{\mathbf{U}}_y(\bar{x}_k, \cdot) = \mathbf{A}\tilde{\mathbf{u}}(\bar{x}_k, \cdot)$, where $\mathbf{A} = \bar{\mathbf{T}}\mathbf{T}^{-1}$ with size $N_2 \times N_2$. Thus, for $k \in \{1, 2, 3, ..., N_1\}$,

$$\begin{bmatrix} \tilde{\mathbf{U}}_y(\bar{x}_1, \cdot) \\ \tilde{\mathbf{U}}_y(\bar{x}_2, \cdot) \\ \vdots \\ \tilde{\mathbf{U}}_y(\bar{x}_{N_1}, \cdot) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & 0 & \cdots & 0 \\ 0 & \mathbf{A} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{A} \end{bmatrix}}_{N_1 \text{ blocks}} \begin{bmatrix} \tilde{\mathbf{u}}(\bar{x}_1, \cdot) \\ \tilde{\mathbf{u}}(\bar{x}_2, \cdot) \\ \vdots \\ \tilde{\mathbf{u}}(\bar{x}_{N_1}, \cdot) \end{bmatrix}, \tag{7}$$

or $\tilde{\mathbf{U}}_y = \tilde{\mathbf{A}}_y\tilde{\mathbf{u}}$. Let $\mathbf{P}$ be a transformation matrix defined by

$$\mathbf{P}_{mn} = \begin{cases} 1 & ; \begin{cases} m = N_1 \times (j-1) + i \\ n = N_2 \times (i-1) + j \end{cases} ; \begin{cases} i \in \{1, 2, 3, ..., N_1\} \\ j \in \{1, 2, 3, ..., N_2\} \end{cases}, \\ 0 & ; \text{ otherwise.} \end{cases}$$

Define $\mathbf{U}_y := \mathbf{P}\tilde{\mathbf{U}}_y$ and $\mathbf{u} := \mathbf{P}\tilde{\mathbf{u}}$. With the definition of $\mathbf{P}$, the matrices $\mathbf{U}_y$ and $\mathbf{u}$ are now in the global numbering system. Note that, $\mathbf{P}^{-1} = \mathbf{P}^T$. Therefore, the integration matrix with respect to y in the global numbering system can be expressed as $\mathbf{A}_y = \mathbf{P}\tilde{\mathbf{A}}_y\mathbf{P}^{-1}$. Thus, $\mathbf{U}_y = \mathbf{A}_y\mathbf{u}$. The matrix $\mathbf{A}_y$ is called *the first order integration matrix with respect to y* for our modified FIM in two dimension.

Let $U_x^{(2)}$ denote the double layer integration with respect to x in the global numbering system. Let $\bar{y}_s$ be fixed, by using (5), we have

$$U_x^{(2)}(\bar{x}_k, \bar{y}_s) = \int_{-1}^{\bar{x}_k} \int_{-1}^{\xi_2} u(\xi_1, \bar{y}_s)\, d\xi_1 d\xi_2 = \sum_{i=1}^{N_1} a_{ki}^{(2)} u(\bar{x}_i, \bar{y}_s)$$

for $k \in \{1, 2, 3, ..., N_1\}$ or $\mathbf{U}_x^{(2)}(\cdot, \bar{y}_s) = \mathbf{A}^2\mathbf{u}(\cdot, \bar{y}_s)$. Thus, for $s \in \{1, 2, 3, ..., N_2\}$,

$$\begin{bmatrix} \mathbf{U}_x^{(2)}(\cdot, \bar{y}_1) \\ \mathbf{U}_x^{(2)}(\cdot, \bar{y}_2) \\ \vdots \\ \mathbf{U}_x^{(2)}(\cdot, \bar{y}_{N_2}) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A}^2 & 0 & \cdots & 0 \\ 0 & \mathbf{A}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{A}^2 \end{bmatrix}}_{N_2 \text{ blocks}} \begin{bmatrix} \mathbf{u}(\cdot, \bar{y}_1) \\ \mathbf{u}(\cdot, \bar{y}_2) \\ \vdots \\ \mathbf{u}(\cdot, \bar{y}_{N_2}) \end{bmatrix},$$

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which can be written in the matrix form as $\mathbf{U}_x^{(2)} = \mathbf{A}_x^{(2)}\mathbf{u} = \mathbf{A}_x^2\mathbf{u}$.

By using the same arguments, the double layer integration with respect to y, $U_y^{(2)}$, can be written in the matrix form as $\tilde{\mathbf{U}}_y^{(2)}(\bar{x}_k, \cdot) = \mathbf{A}^2\tilde{\mathbf{u}}(\bar{x}_k, \cdot)$. Thus, for $k \in \{1, 2, 3, ..., N_1\}$,

$$\begin{bmatrix} \tilde{\mathbf{U}}_y^{(2)}(\bar{x}_1, \cdot) \\ \tilde{\mathbf{U}}_y^{(2)}(\bar{x}_2, \cdot) \\ \vdots \\ \tilde{\mathbf{U}}_y^{(2)}(\bar{x}_{N_1}, \cdot) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A}^2 & 0 & \cdots & 0 \\ 0 & \mathbf{A}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{A}^2 \end{bmatrix}}_{N_1 \text{ blocks}} \begin{bmatrix} \tilde{\mathbf{u}}(\bar{x}_1, \cdot) \\ \tilde{\mathbf{u}}(\bar{x}_2, \cdot) \\ \vdots \\ \tilde{\mathbf{u}}(\bar{x}_{N_1}, \cdot) \end{bmatrix},$$

which can be written in the matrix form as $\tilde{\mathbf{U}}_y^{(2)} = \tilde{\mathbf{A}}_y^{(2)}\tilde{\mathbf{u}} = \tilde{\mathbf{A}}_y^2\tilde{\mathbf{u}}$. Then we obtain $\mathbf{U}_y^{(2)} = \mathbf{A}_y^2\mathbf{u}$, where $\mathbf{A}_y = \mathbf{P}\tilde{\mathbf{A}}_y\mathbf{P}^{-1}$.

Remark 2.3 For $m, n \in \mathbb{N}$, the double layer integration can be easily extended to the multi-layer integration in the global numbering system, which can be represented in the matrix form as follows:

(1) the multi-layer integration with respect to the variables x only or y only is

$$\mathbf{U}_x^{(m)} = \mathbf{A}_x^m\mathbf{u} \text{ or } \mathbf{U}_y^{(m)} = \mathbf{A}_y^m\mathbf{u},$$

(2) the multi-layer integration with respect to the variables both x and y is

$$\mathbf{U}_{xy}^{(m,n)} = \mathbf{A}_x^m\mathbf{A}_y^n\mathbf{u} \text{ or } \mathbf{U}_{yx}^{(n,m)} = \mathbf{A}_y^n\mathbf{A}_x^m\mathbf{u},$$

where $\mathbf{A}_x$ and $\mathbf{A}_y$ are the first order integral matrices with respect to the variables x and y, respectively.

3 Solving linear ODEs in one dimension

Consider the linear ODE $Lv(x) = f(x)$ for $x \in (a, b)$, where $L = \alpha_n(x)\frac{d^n}{dx^n} + \alpha_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \alpha_{n-2}(x)\frac{d^{n-2}}{dx^{n-2}} + ... + \alpha_0(x)$, v is the unknown function, $\alpha_i(x)$'s are continuous functions, and f is a given function. Let u be the approximate solution of v as defined in (4). The procedure for solving linear ODEs is as follows:

Step 1. Transforming $x \in [a, b]$ into $\bar{x} \in [-1, 1]$ by using the transformation $\bar{x} = \frac{2x-a-b}{b-a}$. Let $h = \frac{2}{b-a}$. Then, $Lu(x) = f(x)$ for $x \in (a, b)$ becomes

$$\bar{L}u(\bar{x}) = \bar{f}(\bar{x}) \text{ for } \bar{x} \in (-1, 1), \tag{8}$$

where $\bar{L} = h^n\bar{\alpha}_n(\bar{x})\frac{d^n}{d\bar{x}^n} + h^{n-1}\bar{\alpha}_{n-1}(\bar{x})\frac{d^{n-1}}{d\bar{x}^{n-1}} + h^{n-2}\bar{\alpha}_{n-2}(\bar{x})\frac{d^{n-2}}{d\bar{x}^{n-2}} + ... + \bar{\alpha}_0(\bar{x})$, $\bar{f}(\bar{x}) = f(\frac{(b-a)\bar{x}+a+b}{2})$ and $\bar{\alpha}_i(\bar{x}) = \alpha_i(\frac{(b-a)\bar{x}+a+b}{2})$ for $i \in \{0, 1, 2, ..., n\}$.

Step 2. Discretizing $[-1, 1]$ into N nodes. Use the zeros of $T_N(x)$ as defined in (2) to generate the grid points $\{\bar{x}_k\}_{k=1}^N$.

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Step 3. Eliminating all derivatives of (8) by taking n layers integration from -1 to $\bar{x}_k$ on both sides of (8) and use the integration by parts, we have

$$
\begin{aligned}
& h^n\left[\sum_{i=0}^{n}(-1)^i\binom{n}{i}\int_{-1}^{\bar{x}_k}\cdots\int_{-1}^{\eta_2}\bar{\alpha}_n^{(i)}u\,d\eta_1\ldots d\eta_i\right] \\
& + h^{n-1}\int_{-1}^{\bar{x}_k}\left[\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\int_{-1}^{\xi_n}\cdots\int_{-1}^{\eta_2}\bar{\alpha}_{n-1}^{(i)}u\,d\eta_1\ldots d\eta_i\right]d\xi_n \\
& + h^{n-2}\int_{-1}^{\bar{x}_k}\int_{-1}^{\xi_n}\left[\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}\int_{-1}^{\xi_{n-1}}\cdots\int_{-1}^{\eta_2}\bar{\alpha}_{n-2}^{(i)}u\,d\eta_1\ldots d\eta_i\right]d\xi_{n-1}d\xi_n \\
& \vdots \\
& + \int_{-1}^{\bar{x}_k}\cdots\int_{-1}^{\xi_2}\bar{\alpha}_0(\xi_1)u(\xi_1)\,d\xi_1\ldots d\xi_n + d_1\frac{\bar{x}_k^{n-1}}{(n-1)!} + d_2\frac{\bar{x}_k^{n-2}}{(n-2)!} + \ldots + d_n \\
& = \int_{-1}^{\bar{x}_k}\cdots\int_{-1}^{\xi_2}\bar{f}(\xi_1)\,d\xi_1\ldots d\xi_n, \qquad (9)
\end{aligned}
$$

where $d_1, d_2, d_3, ..., d_n$ are the arbitrary constants of integration.

Step 4. Transforming (9) into the matrix form by using the idea described in Section 2.1,

$$
\begin{aligned}
& h^n\left[\sum_{i=0}^{n}(-1)^i\binom{n}{i}\mathbf{A}^i\bar{\boldsymbol{\alpha}}_n^{(i)}\mathbf{u}\right] \\
& + h^{n-1}\mathbf{A}^1\left[\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\mathbf{A}^i\bar{\boldsymbol{\alpha}}_{n-1}^{(i)}\mathbf{u}\right] \\
& + h^{n-2}\mathbf{A}^2\left[\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}\mathbf{A}^i\bar{\boldsymbol{\alpha}}_{n-2}^{(i)}\mathbf{u}\right] \\
& \vdots \\
& + \mathbf{A}^n\bar{\boldsymbol{\alpha}}_0^{(0)}\mathbf{u} + d_1\bar{\mathbf{x}}_{n-1} + d_2\bar{\mathbf{x}}_{n-2} + \ldots + d_n\bar{\mathbf{x}}_0 = \mathbf{A}^n\bar{\mathbf{f}}.
\end{aligned}
$$

Let $\mathbf{K} = \sum_{j=0}^{n}\left[h^{n-j}\sum_{i=0}^{n-j}(-1)^i\binom{n-j}{i}\mathbf{A}^{i+j}\bar{\boldsymbol{\alpha}}_{n-j}^{(i)}\right]$, then we have

$$
\mathbf{K}\mathbf{u} + d_1\bar{\mathbf{x}}_{n-1} + d_2\bar{\mathbf{x}}_{n-2} + \ldots + d_n\bar{\mathbf{x}}_0 = \mathbf{A}^n\bar{\mathbf{f}}, \qquad (10)
$$

where $\mathbf{A} = \bar{\mathbf{T}}\mathbf{T}^{-1}$, $\bar{\boldsymbol{\alpha}}_{n-j}^{(i)} = \text{diag}(\bar{\alpha}_{n-j}^{(i)}(\bar{x}_1), \bar{\alpha}_{n-j}^{(i)}(\bar{x}_2), \bar{\alpha}_{n-j}^{(i)}(\bar{x}_3), ..., \bar{\alpha}_{n-j}^{(i)}(\bar{x}_N))$ for $i, j \in \{0, 1, 2, ..., n\}$, $\bar{\mathbf{f}} = [\bar{f}(\bar{x}_1), \bar{f}(\bar{x}_2), \bar{f}(\bar{x}_3), ..., \bar{f}(\bar{x}_N)]^T$, $\bar{\mathbf{x}}_{n-i} = \frac{1}{(n-i)!}[\bar{x}_1^{n-i}, \bar{x}_2^{n-i}, \bar{x}_3^{n-i}, ..., \bar{x}_N^{n-i}]^T$ for $i \in \{1, 2, 3, ..., n\}$.

Step 5. Changing the initial or boundary conditions into the vector form by using the linear combination of Chebyshev polynomials at the boundary ends $x = \pm 1$ and use (3). Let $p \in \mathbb{N}$, we have

$$
u(\pm 1) = \sum_{n=0}^{N-1}c_nT_n(\pm 1) = \sum_{n=0}^{N-1}c_n(\pm 1)^n = \mathbf{t}_0\mathbf{c} = \mathbf{t}_0\mathbf{T}^{-1}\mathbf{u} \text{ and}
$$

$$
u^{(p)}(\pm 1) = \sum_{n=0}^{N-1}c_nT_n^{(p)}(\pm 1) = \sum_{n=0}^{N-1}c_n(\pm 1)^{p+n}\prod_{k=0}^{p-1}\frac{n^2-k^2}{2k+1} = \mathbf{t}_p\mathbf{c} = \mathbf{t}_p\mathbf{T}^{-1}\mathbf{u},
$$

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where

$$\mathbf{t}_0 = \begin{bmatrix} (\pm 1)^0 \\ (\pm 1)^1 \\ (\pm 1)^2 \\ \vdots \\ (\pm 1)^{N-1} \end{bmatrix}^T \quad \text{and} \quad \mathbf{t}_p = \begin{bmatrix} (\pm 1)^{p+0} \prod_{k=0}^{p-1} \frac{-k^2}{2k+1} \\ (\pm 1)^{p+1} \prod_{k=0}^{p-1} \frac{1-k^2}{2k+1} \\ (\pm 1)^{p+2} \prod_{k=0}^{p-1} \frac{4-k^2}{2k+1} \\ \vdots \\ (\pm 1)^{p+N-1} \prod_{k=0}^{p-1} \frac{(N-1)^2-k^2}{2k+1} \end{bmatrix}^T.$$

Suppose boundary conditions are $u(\pm 1) = b_0, u'(\pm 1) = b_1, ..., u^{(n-1)}(\pm 1) = b_{n-1}$, which can be represented in the vector form as

$$u(\pm 1) = \mathbf{t}_0 \mathbf{T}^{-1} \mathbf{u} = b_0, u'(\pm 1) = \mathbf{t}_1 \mathbf{T}^{-1} \mathbf{u} = b_1, ..., u^{(n-1)}(\pm 1) = \mathbf{t}_{n-1} \mathbf{T}^{-1} \mathbf{u} = b_{n-1}.$$

Note that, the left and right boundary conditions are defined by $u^{(k)}(-1) = \mathbf{t}_{kl} \mathbf{T}^{-1} \mathbf{u}$ and $u^{(k)}(1) = \mathbf{t}_{kr} \mathbf{T}^{-1} \mathbf{u}$, where $\mathbf{t}_{kl}$ and $\mathbf{t}_{kr}$ are the row vector $\mathbf{t}_k$ for $k \in \mathbb{N} \cup \{0\}$ that their elements are substituted by -1 and 1, respectively.

Step 6. Use (10) and the boundary conditions in Step 5, including $\mathbf{t}_0 \mathbf{T}^{-1} \mathbf{u} = b_0$, $\mathbf{t}_1 \mathbf{T}^{-1} \mathbf{u} = b_1$, ..., $\mathbf{t}_{n-1} \mathbf{T}^{-1} \mathbf{u} = b_{n-1}$ to construct the linear system. We obtain the linear system in the matrix form as

$$\left[\begin{array}{c|cccc} \mathbf{K} & \bar{\mathbf{x}}_{n-1} & \bar{\mathbf{x}}_{n-2} & \cdots & \bar{\mathbf{x}}_0 \\ \hline \mathbf{t}_0 \mathbf{T}^{-1} & 0 & 0 & \cdots & 0 \\ \mathbf{t}_1 \mathbf{T}^{-1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{t}_{n-1} \mathbf{T}^{-1} & 0 & 0 & \cdots & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \hline d_1 \\ d_2 \\ \vdots \\ d_n \end{array}\right] = \left[\begin{array}{c} \mathbf{A}^n \bar{\mathbf{f}} \\ \hline b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{array}\right], \tag{11}$$

which we denote (11) by the block matrix

$$\begin{bmatrix} \mathbf{K} & \mathbf{Q} \\ \mathbf{R} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} \mathbf{z} \\ \mathbf{b} \end{bmatrix}.$$

The linear system (11) can be solved by using the Schur complement method [6], which we assume $\mathbf{K}$ and $\mathbf{R}\mathbf{K}^{-1}\mathbf{Q}$ are nonsingular matrices. Then we have

$$\mathbf{u} = \mathbf{K}^{-1} \left[\mathbf{z} - \mathbf{Q} \left(\mathbf{R} \mathbf{K}^{-1} \mathbf{Q} \right)^{-1} \left(\mathbf{R} \mathbf{K}^{-1} \mathbf{z} - \mathbf{b} \right) \right]. \tag{12}$$

From (12), we have the solution $u(\bar{x})$ for $\bar{x} \in [-1, 1]$. To obtain the approximate solution $u(x)$ for $x \in [a, b]$, we use the transformation $x = \frac{1}{2}[(b-a)\bar{x} + a + b]$.

Example 3.1 Consider the boundary value problem

$$\frac{d^2 u}{dx^2} - u = x \text{ for } x \in (0, 1), u(0) = 0, u(1) = 1.$$

The exact solution is $u^*(x) = \frac{2\sinh(x)}{\sinh(1)} - x$. By using the procedure described above, the problem is transformed to

$$4u''(\bar{x}) - u(\bar{x}) = \bar{f}(\bar{x}) \text{ for } \bar{x} \in (0, 1), u(-1) = 0, u(1) = 1,$$

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where $\bar{f}(\bar{x}) = \frac{\bar{x}+1}{2}$. Taking the double layer integration on both sides of the ODE, we obtain $4\mathbf{u} - \mathbf{A}^2\mathbf{u} + d_1\bar{\mathbf{x}}_1 + d_2\bar{\mathbf{x}}_0 = \mathbf{A}^2\bar{\mathbf{f}}$. With the boundary conditions, we have $u(-1) = \mathbf{t}_{0l}\mathbf{T}^{-1}\mathbf{u} = 0$ and $u(1) = \mathbf{t}_{0r}\mathbf{T}^{-1}\mathbf{u} = 1$. Thus, we can construct the linear system (11) as

$$\left[\begin{array}{c|cc} 4\mathbf{I} - \mathbf{A}^2 & \bar{\mathbf{x}}_1 & \bar{\mathbf{x}}_0 \\ \hline \mathbf{t}_{0l}\mathbf{T}^{-1} & 0 & 0 \\ \mathbf{t}_{0r}\mathbf{T}^{-1} & 0 & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \hline d_1 \\ d_2 \end{array}\right] = \left[\begin{array}{c} \mathbf{A}^2\bar{\mathbf{f}} \\ \hline 0 \\ 1 \end{array}\right].$$

The average relative error is shown in Table 1.

N	FDM	FIM(TPZ)	FIM(SIM)	FIM(CBS)
6	6.6245×10^{-4}	1.3409×10^{-3}	1.2079×10^{-3}	1.0024×10^{-5}
8	3.5884×10^{-4}	7.2209×10^{-4}	3.9900×10^{-4}	1.5034×10^{-8}
10	2.2389×10^{-4}	4.4944×10^{-4}	1.8455×10^{-4}	1.3094×10^{-11}
12	1.5273×10^{-4}	3.0621×10^{-4}	1.0390×10^{-4}	1.4481×10^{-14}
14	1.1075×10^{-4}	2.2189×10^{-4}	6.6077×10^{-5}	8.5544×10^{-15}

Table 1: Average relative error for Example 3.1

Example 3.2 Consider the boundary value problem

$$\frac{d^3u}{dx^3} + x^2\frac{d^2u}{dx^2} - u = e^x(x^3 + 2x^2 + 3) \text{ for } x \in (1, 3),$$

and $u(1) = e, u(3) = 3e^3, u'(1) = 2e$. The exact solution is $u^*(x) = e^{x+\ln x}$. By using our procedure, the problem is transformed to

$$u'''(\bar{x}) + \bar{\alpha}u''(\bar{x}) - u(\bar{x}) = \bar{f}(\bar{x}), u(-1) = e, u(1) = 3e^3, u'(-1) = 2e,$$

where $\bar{\alpha}(\bar{x}) = (\bar{x}+2)^2$ and $\bar{f}(\bar{x}) = e^{\bar{x}+2}[(\bar{x}+2)^3 + 2(\bar{x}+2)^2 + 3]$. The transformed problem can be written in the matrix form as $\mathbf{K}\mathbf{u} + d_1\bar{\mathbf{x}}_2 + d_2\bar{\mathbf{x}}_1 + d_3\bar{\mathbf{x}}_0 = \mathbf{A}^3\bar{\mathbf{f}}$, where $\mathbf{K} = \mathbf{I} + \mathbf{A}\bar{\boldsymbol{\alpha}}^{(0)} - 2\mathbf{A}^2\bar{\boldsymbol{\alpha}}^{(1)} + \mathbf{A}^3\bar{\boldsymbol{\alpha}}^{(2)} - \mathbf{A}^3$. With the boundary conditions, $u(-1) = \mathbf{t}_{0l}\mathbf{T}^{-1}\mathbf{u} = e$, $u(1) = \mathbf{t}_{0r}\mathbf{T}^{-1}\mathbf{u} = 3e^3$ and $u'(-1) = \mathbf{t}_{1l}\mathbf{T}^{-1}\mathbf{u} = 2e$. We obtain the linear system

$$\left[\begin{array}{c|ccc} \mathbf{K} & \bar{\mathbf{x}}_2 & \bar{\mathbf{x}}_1 & \bar{\mathbf{x}}_0 \\ \hline \mathbf{t}_{0l}\mathbf{T}^{-1} & 0 & 0 & 0 \\ \mathbf{t}_{0r}\mathbf{T}^{-1} & 0 & 0 & 0 \\ \mathbf{t}_{1l}\mathbf{T}^{-1} & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \hline d_1 \\ d_2 \\ d_3 \end{array}\right] = \left[\begin{array}{c} \mathbf{A}^3\bar{\mathbf{f}} \\ \hline e \\ 3e^3 \\ 2e \end{array}\right].$$

The average relative error as shown in Table 2.

Example 3.3 Consider the boundary value problem in [4]

$$\frac{d^4u}{dx^4} + u = 1 \text{ for } x \in (0, 1), u(0) = u(1) = u''(0) = u''(1) = 0.$$

The exact solution is

$$u^*(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi x]}{(2n-1)\left[(2n-1)^4\pi^4 + 1\right]}.$$

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N	FDM	FIM(TPZ)	FIM(SIM)	FIM(CBS)
6	1.7380×10^{-1}	5.4341×10^{-2}	2.3800×10^{-3}	1.4554×10^{-3}
8	1.2701×10^{-1}	2.9003×10^{-2}	9.0303×10^{-3}	1.8353×10^{-5}
10	9.4335×10^{-2}	1.8015×10^{-2}	4.2053×10^{-3}	1.0181×10^{-7}
12	7.1429×10^{-2}	1.2272×10^{-3}	2.2325×10^{-3}	3.3192×10^{-10}
14	5.5446×10^{-2}	8.8953×10^{-3}	1.3012×10^{-4}	7.5743×10^{-13}

Table 2: Average relative error for Example 3.2

By using our procedure, the problem is transformed to

$$16u^{(4)}(\bar{x}) + u(\bar{x}) = 1, u(-1) = u(1) = u''(-1) = u''(1) = 0.$$

By taking four layer integration on both sides of the ODE, we obtain $16\mathbf{u} + \mathbf{A}^4\mathbf{u} + d_1\bar{\mathbf{x}}_3 + d_2\bar{\mathbf{x}}_2 + d_3\bar{\mathbf{x}}_1 + d_4\bar{\mathbf{x}}_0 = \mathbf{A}^4\bar{\mathbf{x}}_0$. With the boundary conditions, $u(-1) = \mathbf{t}_{0l}\mathbf{T}^{-1}\mathbf{u} = 0$, $u(1) = \mathbf{t}_{0r}\mathbf{T}^{-1}\mathbf{u} = 0$, $u''(-1) = \mathbf{t}_{2l}\mathbf{T}^{-1}\mathbf{u} = 0$ and $u''(1) = \mathbf{t}_{2r}\mathbf{T}^{-1}\mathbf{u} = 0$. We have the linear system

$$\left[\begin{array}{c|cccc} 16\mathbf{I} + \mathbf{A}^4 & \bar{\mathbf{x}}_3 & \bar{\mathbf{x}}_2 & \bar{\mathbf{x}}_1 & \bar{\mathbf{x}}_0 \\ \hline \mathbf{t}_{0l}\mathbf{T}^{-1} & 0 & 0 & 0 & 0 \\ \mathbf{t}_{0r}\mathbf{T}^{-1} & 0 & 0 & 0 & 0 \\ \mathbf{t}_{2l}\mathbf{T}^{-1} & 0 & 0 & 0 & 0 \\ \mathbf{t}_{2r}\mathbf{T}^{-1} & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \hline d_1 \\ d_2 \\ d_3 \\ d_4 \end{array}\right] = \left[\begin{array}{c} \mathbf{A}^4\bar{\mathbf{x}}_0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right].$$

The Table 3 shows the average absolute error for Example 3.3.

N	FDM	FIM(TPZ)	FIM(SIM)	FIM(CBS)
6	2.1523×10^{-4}	4.3201×10^{-4}	1.3239×10^{-4}	7.7252×10^{-6}
8	1.1775×10^{-4}	2.3594×10^{-4}	4.0609×10^{-5}	2.4808×10^{-7}
10	7.3896×10^{-5}	1.4796×10^{-4}	1.6204×10^{-5}	5.0072×10^{-11}
12	5.0600×10^{-5}	1.0128×10^{-4}	7.6577×10^{-6}	4.4890×10^{-13}
14	3.6790×10^{-5}	7.3618×10^{-5}	4.0681×10^{-6}	7.2176×10^{-14}

Table 3: Average absolute error for Example 3.3

Figure 2 shows the numerical solutions $\mathbf{u}$ of Examples 3.1 - 3.3 by our modified FIM compared with the analytical solutions.

4 Solving Linear PDEs in two dimension

Let $\Omega = [a, b] \times [c, d]$ and consider the linear PDE

$$\alpha_1 \frac{\partial^2 u}{\partial x^2} + \alpha_2 \frac{\partial^2 u}{\partial y^2} + \alpha_3 \frac{\partial^2 u}{\partial x \partial y} + \alpha_4 \frac{\partial u}{\partial x} + \alpha_5 \frac{\partial u}{\partial y} + \alpha_6 u = \beta,\ (x, y) \in \Omega,$$

under boundary condition $u(x, y) = \omega$, $(x, y) \in \partial\Omega$, where α_1, α_2, α_3, α_4, α_5, α_6, β and ω are given functions of x and y. The procedure is given as follows:

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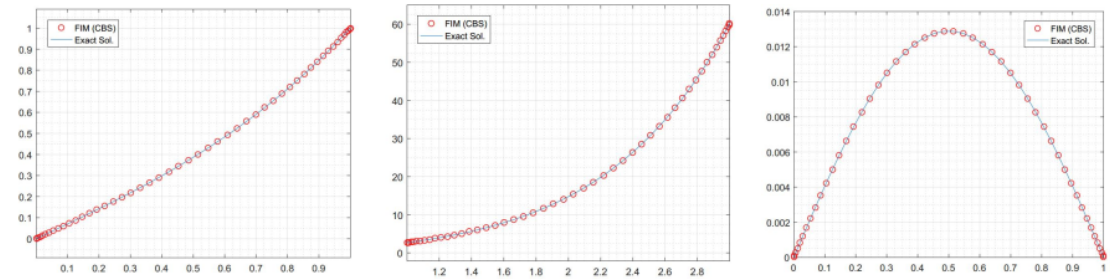

(a) Example 3.1 (b) Example 3.2 (c) Example 3.3

Figure 2: The graphs of solutions in Examples 3.1-3.3

Step 1. Transforming $\Omega = [a, b] \times [c, d]$ into $\bar{\Omega} = [-1, 1] \times [-1, 1]$ by using $\bar{x} = \frac{2x-a-b}{b-a}$ and $\bar{y} = \frac{2y-c-d}{d-c}$. Let $\hat{h} = \frac{2}{b-a}$ and $\hat{k} = \frac{2}{d-c}$. Then, we have

$$\hat{h}^2 \bar{\alpha}_1 \frac{\partial^2 u}{\partial \bar{x}^2} + \hat{k}^2 \bar{\alpha}_2 \frac{\partial^2 u}{\partial \bar{y}^2} + \hat{h}\hat{k}\bar{\alpha}_3 \frac{\partial^2 u}{\partial \bar{x} \partial \bar{y}} + \hat{h}\bar{\alpha}_4 \frac{\partial u}{\partial \bar{x}} + \hat{k}\bar{\alpha}_5 \frac{\partial u}{\partial \bar{y}} + \bar{\alpha}_6 u = \bar{\beta} \tag{13}$$

for $(\bar{x}, \bar{y}) \in \bar{\Omega}$, under boundary condition $u(\bar{x}, \bar{y}) = \bar{\omega}$, $(\bar{x}, \bar{y}) \in \partial\bar{\Omega}$.

Step 2. Discretizing $[-1, 1] \times [-1, 1]$ into $N_1 \times N_2$ nodes. Let $\{\bar{x}_k\}_{k=1}^{N_1}$ and $\{\bar{y}_s\}_{s=1}^{N_2}$ be the grid points generated by (2) along x and y directions, respectively, that are

$$\bar{x}_k = \cos\left[\frac{(2k-1)\pi}{2N_1}\right], \quad k \in \{1, 2, 3, ..., N_1\},$$

$$\bar{y}_s = \cos\left[\frac{(2s-1)\pi}{2N_2}\right], \quad s \in \{1, 2, 3, ..., N_2\}.$$

Then, the number of total grid points in the global system is $M = N_1 \times N_2$.

Step 3. Eliminating all derivatives of (13) by taking four layer integration, which integrates twice over both x and y, and use integration by parts, we have

$$\begin{aligned}
&\hat{h}^2 \int_{-1}^{\bar{y}_s}\int_{-1}^{\eta_2}\left[\bar{\alpha}_1 u - 2\int_{-1}^{\bar{x}_k} \bar{\alpha}_{1\bar{x}} u\, d\xi_2 + \int_{-1}^{\bar{x}_k}\int_{-1}^{\xi_2} \bar{\alpha}_{1\bar{x}\bar{x}} u\, d\xi_1 d\xi_2\right] d\eta_1 d\eta_2 \\
&+ \hat{k}^2 \int_{-1}^{\bar{x}_k}\int_{-1}^{\xi_2}\left[\bar{\alpha}_2 u - 2\int_{-1}^{\bar{y}_s} \bar{\alpha}_{2\bar{y}} u\, d\eta_2 + \int_{-1}^{\bar{y}_s}\int_{-1}^{\eta_2} \bar{\alpha}_{2\bar{y}\bar{y}} u\, d\eta_1 d\eta_2\right] d\xi_1 d\xi_2 \\
&+ \hat{h}\hat{k} \int_{-1}^{\bar{y}_s}\int_{-1}^{\bar{x}_k}\left[\bar{\alpha}_3 u - \int_{-1}^{\xi_2} \bar{\alpha}_{3\bar{x}} u\, d\xi_1 - \int_{-1}^{\eta_2} \bar{\alpha}_{3\bar{y}} u\, d\eta_1 + \int_{-1}^{\eta_2}\int_{-1}^{\xi_2} \bar{\alpha}_{3\bar{x}\bar{y}} u\, d\xi_1 d\eta_1\right] d\xi_2 d\eta_2 \\
&+ \hat{h} \int_{-1}^{\bar{y}_s}\int_{-1}^{\eta_2}\int_{-1}^{\bar{x}_k}\left[\bar{\alpha}_4 u - \int_{-1}^{\xi_2} \bar{\alpha}_{4\bar{x}} u\, d\xi_1\right] d\xi_2 d\eta_1 d\eta_2 \\
&+ \hat{k} \int_{-1}^{\bar{y}_s}\int_{-1}^{\bar{x}_k}\int_{-1}^{\xi_2}\left[\bar{\alpha}_5 u - \int_{-1}^{\eta_2} \bar{\alpha}_{5\bar{y}} u\, d\eta_1\right] d\xi_1 d\xi_2 d\eta_2 \\
&+ \int_{-1}^{\bar{y}_s}\int_{-1}^{\eta_2}\int_{-1}^{\bar{x}_k}\int_{-1}^{\xi_2} \bar{\alpha}_6 u\, d\xi_1 d\xi_2 d\eta_1 d\eta_2 + \bar{x}_k f_0(\bar{y}_s) + f_1(\bar{y}_s) + \bar{y}_s g_0(\bar{x}_k) + g_1(\bar{x}_k) \\
&= \int_{-1}^{\bar{y}_s}\int_{-1}^{\eta_2}\int_{-1}^{\bar{x}_k}\int_{-1}^{\xi_2} \bar{\beta}\, d\xi_1 d\xi_2 d\eta_1 d\eta_2,
\end{aligned} \tag{14}$$

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where $f_0(\bar{y}_s), f_1(\bar{y}_s), g_0(\bar{x}_k)$ and $g_1(\bar{x}_k)$ are the arbitrary functions of integration assumed to be approximated by Chebyshev interpolating polynomials,

$$g_r(\bar{x}_k) = \sum_{i=0}^{P_1-1} g_i^{(r)} T_i(\bar{x}_k) \text{ and } f_r(\bar{y}_s) = \sum_{j=0}^{P_2-1} f_j^{(r)} T_j(\bar{y}_s) \tag{15}$$

for $r \in \{0, 1\}$, where $g_0^{(r)}, g_1^{(r)}, g_2^{(r)}, ..., g_{P_1-1}^{(r)}$ and $f_0^{(r)}, f_1^{(r)}, f_2^{(r)}, ..., f_{P_2-1}^{(r)}$ are the unknown values of these interpolated points. Note that, the number of these values $(2P_1 + 2P_2)$ should be equal to the number of boundary points $(2N_1 + 2N_2)$ in order that a linear system is solvable. Thus, we select $P_1 = N_1$ and $P_2 = N_2$.

Step 4. Transforming (14) into the matrix form by using the idea described in Section 2.2, which the index is arranged in the global numbering system as

$$\begin{aligned}
&\hat{h}^2\mathbf{A}_y^2(\bar{\boldsymbol{\alpha}}_1\mathbf{u} - 2\mathbf{A}_x\bar{\boldsymbol{\alpha}}_{1\bar{x}}\mathbf{u} + \mathbf{A}_x^2\bar{\boldsymbol{\alpha}}_{1\bar{x}\bar{x}}\mathbf{u}) \\
&+ \hat{k}^2\mathbf{A}_x^2(\bar{\boldsymbol{\alpha}}_2\mathbf{u} - 2\mathbf{A}_y\bar{\boldsymbol{\alpha}}_{2\bar{y}}\mathbf{u} + \mathbf{A}_y^2\bar{\boldsymbol{\alpha}}_{2\bar{y}\bar{y}}\mathbf{u}) \\
&+ \hat{h}\hat{k}\mathbf{A}_x\mathbf{A}_y(\bar{\boldsymbol{\alpha}}_3\mathbf{u} - \mathbf{A}_x\bar{\boldsymbol{\alpha}}_{3\bar{x}}\mathbf{u} - \mathbf{A}_y\bar{\boldsymbol{\alpha}}_{3\bar{y}}\mathbf{u} + \mathbf{A}_x\mathbf{A}_y\bar{\boldsymbol{\alpha}}_{3\bar{x}\bar{y}}\mathbf{u}) \\
&+ \hat{h}\mathbf{A}_x\mathbf{A}_y^2(\bar{\boldsymbol{\alpha}}_4\mathbf{u} - \mathbf{A}_x\bar{\boldsymbol{\alpha}}_{4\bar{x}}\mathbf{u}) + \hat{k}\mathbf{A}_x^2\mathbf{A}_y(\bar{\boldsymbol{\alpha}}_5\mathbf{u} - \mathbf{A}_y\bar{\boldsymbol{\alpha}}_{5\bar{y}}\mathbf{u}) \\
&+ \mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\alpha}}_6\mathbf{u} + \mathbf{X}\boldsymbol{\Phi}_y\mathbf{f}_0 + \boldsymbol{\Phi}_y\mathbf{f}_1 + \mathbf{Y}\boldsymbol{\Phi}_x\mathbf{g}_0 + \boldsymbol{\Phi}_x\mathbf{g}_1 = \mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\beta}}.
\end{aligned}$$

Let $\mathbf{K} = \hat{h}^2\mathbf{A}_y^2(\bar{\boldsymbol{\alpha}}_1 - 2\mathbf{A}_x\bar{\boldsymbol{\alpha}}_{1\bar{x}} + \mathbf{A}_x^2\bar{\boldsymbol{\alpha}}_{1\bar{x}\bar{x}}) + \hat{k}^2\mathbf{A}_x^2(\bar{\boldsymbol{\alpha}}_2 - 2\mathbf{A}_y\bar{\boldsymbol{\alpha}}_{2\bar{x}} + \mathbf{A}_y^2\bar{\boldsymbol{\alpha}}_{2\bar{y}\bar{y}}) + \hat{h}\hat{k}\mathbf{A}_x\mathbf{A}_y(\bar{\boldsymbol{\alpha}}_3 - \mathbf{A}_x\bar{\boldsymbol{\alpha}}_{3\bar{x}} - \mathbf{A}_y\bar{\boldsymbol{\alpha}}_{3\bar{y}} + \mathbf{A}_x\mathbf{A}_y\bar{\boldsymbol{\alpha}}_{3\bar{x}\bar{y}}) + \hat{h}\mathbf{A}_x\mathbf{A}_y^2(\bar{\boldsymbol{\alpha}}_4 - \mathbf{A}_x\bar{\boldsymbol{\alpha}}_{4\bar{x}}) + \hat{k}\mathbf{A}_x^2\mathbf{A}_y(\bar{\boldsymbol{\alpha}}_5 - \mathbf{A}_y\bar{\boldsymbol{\alpha}}_{5\bar{y}}) + \mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\alpha}}_6$. Then, we have

$$\mathbf{K}\mathbf{u} + \mathbf{X}\boldsymbol{\Phi}_y\mathbf{f}_0 + \boldsymbol{\Phi}_y\mathbf{f}_1 + \mathbf{Y}\boldsymbol{\Phi}_x\mathbf{g}_0 + \boldsymbol{\Phi}_x\mathbf{g}_1 = \mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\beta}}, \tag{16}$$

where $\mathbf{g}_r = [g_0^{(r)}, g_1^{(r)}, ..., g_{P_1-1}^{(r)}]^T$ and $\mathbf{f}_r = [f_0^{(r)}, f_1^{(r)}, ..., f_{P_2-1}^{(r)}]^T$ for $r \in \{0, 1\}$, $\mathbf{A}_x$ and $\mathbf{A}_y$ as defined in (6) and (7). The matrices $\mathbf{X}$ and $\mathbf{Y}$ are defined by

$$\mathbf{X} = \underbrace{\begin{bmatrix} \mathbf{X}_0 & 0 & \cdots & 0 \\ 0 & \mathbf{X}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{X}_0 \end{bmatrix}}_{N_2 \text{ blocks}} \quad \text{when} \quad \mathbf{X}_0 = \begin{bmatrix} \bar{x}_1 & 0 & \cdots & 0 \\ 0 & \bar{x}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{x}_{N_1} \end{bmatrix},$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{Y}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{Y}_{N_2} \end{bmatrix} \quad \text{when} \quad \mathbf{Y}_s = \underbrace{\begin{bmatrix} \bar{y}_s & 0 & \cdots & 0 \\ 0 & \bar{y}_s & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{y}_s \end{bmatrix}}_{N_1 \times N_1},$$

for $s \in \{1, 2, 3, ..., N_2\}$.

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Let $(\hat{x}_m, \hat{y}_m) = (\mathbf{X}_{mm}, \mathbf{Y}_{mm})$ for $m \in \{1, 2, 3, ..., M\}$, where $\mathbf{X}_{mm}$ and $\mathbf{Y}_{mm}$ are the entries in the m^{th} row and m^{th} column of matrices $\mathbf{X}$ and $\mathbf{Y}$, respectively. For $i \in \{1, 2, 3, 4, 5, 6\}$,

$$\mathbf{u} = [\mathbf{u}(\hat{x}_1, \hat{y}_1), \mathbf{u}(\hat{x}_2, \hat{y}_2), \mathbf{u}(\hat{x}_3, \hat{y}_3), ..., \mathbf{u}(\hat{x}_M, \hat{y}_M)]^T,$$

$$\bar{\boldsymbol{\beta}} = [\bar{\boldsymbol{\beta}}(\hat{x}_1, \hat{y}_1), \bar{\boldsymbol{\beta}}(\hat{x}_2, \hat{y}_2), \bar{\boldsymbol{\beta}}(\hat{x}_3, \hat{y}_3), ..., \bar{\boldsymbol{\beta}}(\hat{x}_M, \hat{y}_M)]^T,$$

$$\bar{\boldsymbol{\alpha}}_i = \begin{bmatrix} \bar{\alpha}_i(\hat{x}_1, \hat{y}_1) & 0 & \cdots & 0 \\ 0 & \bar{\alpha}_i(\hat{x}_2, \hat{y}_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{\alpha}_i(\hat{x}_M, \hat{y}_M) \end{bmatrix}_{M \times M},$$

$$\bar{\boldsymbol{\alpha}}_{i,\cdot} = \begin{bmatrix} \frac{\partial \bar{\alpha}_i}{\partial \cdot}|_{(\hat{x}_1, \hat{y}_1)} & 0 & \cdots & 0 \\ 0 & \frac{\partial \bar{\alpha}_i}{\partial \cdot}|_{(\hat{x}_2, \hat{y}_2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{\partial \bar{\alpha}_i}{\partial \cdot}|_{(\hat{x}_M, \hat{y}_M)} \end{bmatrix}_{M \times M}.$$

From (15), we obtain $\mathbf{\Phi}_x$ and $\mathbf{\Phi}_y$, where

$$\mathbf{\Phi}_x = \begin{bmatrix} T_0(\hat{x}_1) & T_1(\hat{x}_1) & \cdots & T_{P_1-1}(\hat{x}_1) \\ T_0(\hat{x}_2) & T_1(\hat{x}_2) & \cdots & T_{P_1-1}(\hat{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(\hat{x}_M) & T_1(\hat{x}_M) & \cdots & T_{P_1-1}(\hat{x}_M) \end{bmatrix}_{M \times P_1},$$

$$\mathbf{\Phi}_y = \begin{bmatrix} T_0(\hat{y}_1) & T_1(\hat{y}_1) & \cdots & T_{P_2-1}(\hat{y}_1) \\ T_0(\hat{y}_2) & T_1(\hat{y}_2) & \cdots & T_{P_2-1}(\hat{y}_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(\hat{y}_M) & T_1(\hat{y}_M) & \cdots & T_{P_2-1}(\hat{y}_M) \end{bmatrix}_{M \times P_2}.$$

Step 5. The boundary conditions are taken care as follows:

- For the left boundary condition, by letting $u(-1, \bar{y}) = \bar{\omega}_l(\bar{y})$, we have

$$u(-1, \bar{y}_s) = \sum_{n=0}^{N_1-1} c_n T_n(-1) = \mathbf{t}_l \mathbf{c} = \mathbf{t}_l \mathbf{T}^{-1} \mathbf{u}(\cdot, \bar{y}_s) = \bar{\omega}_l(\bar{y}_s)$$

for $s \in \{1, 2, 3, ..., N_2\}$ or

$$\underbrace{\begin{bmatrix} \mathbf{t}_l \mathbf{T}^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{t}_l \mathbf{T}^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{t}_l \mathbf{T}^{-1} \end{bmatrix}}_{N_2 \times M} \begin{bmatrix} \mathbf{u}(\cdot, \bar{y}_1) \\ \mathbf{u}(\cdot, \bar{y}_2) \\ \vdots \\ \mathbf{u}(\cdot, \bar{y}_{N_2}) \end{bmatrix} = \begin{bmatrix} \bar{\omega}_l(\bar{y}_1) \\ \bar{\omega}_l(\bar{y}_2) \\ \vdots \\ \bar{\omega}_l(\bar{y}_{N_2}) \end{bmatrix},$$

which we denote it by $\mathbf{T}_l \mathbf{u} = \bar{\boldsymbol{\omega}}_l$, where $\mathbf{T}^{-1}$ has size $N_1 \times N_1$ and $\mathbf{t}_l = [1, -1, 1, -1, ..., (-1)^{N_1-1}]$.

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- For the right boundary condition, by letting $u(1, \bar{y}) = \bar{\omega}_r(\bar{y})$, we have

$$u(1, \bar{y}_s) = \sum_{n=0}^{N_1-1} c_n T_n(1) = \mathbf{t}_r \mathbf{c} = \mathbf{t}_r \mathbf{T}^{-1} \mathbf{u}(\cdot, \bar{y}_s) = \bar{\omega}_r(\bar{y}_s)$$

for $s \in \{1, 2, 3, ..., N_2\}$ or

$$\underbrace{\begin{bmatrix} \mathbf{t}_r\mathbf{T}^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{t}_r\mathbf{T}^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{t}_r\mathbf{T}^{-1} \end{bmatrix}}_{N_2 \times M} \begin{bmatrix} \mathbf{u}(\cdot, \bar{y}_1) \\ \mathbf{u}(\cdot, \bar{y}_2) \\ \vdots \\ \mathbf{u}(\cdot, \bar{y}_{N_2}) \end{bmatrix} = \begin{bmatrix} \bar{\omega}_r(\bar{y}_1) \\ \bar{\omega}_r(\bar{y}_2) \\ \vdots \\ \bar{\omega}_r(\bar{y}_{N_2}) \end{bmatrix},$$

which we denote it by $\mathbf{T}_r \mathbf{u} = \bar{\boldsymbol{\omega}}_r$, where $\mathbf{T}^{-1}$ has size $N_1 \times N_1$ and $\mathbf{t}_r = [1, 1, 1, ..., (1)^{N_1-1}]$.

- For the bottom boundary condition, by letting $u(\bar{x}, -1) = \bar{\omega}_b(\bar{x})$, we have

$$u(\bar{x}_k, -1) = \sum_{n=0}^{N_2-1} c_n T_n(-1) = \mathbf{t}_b \mathbf{c} = \mathbf{t}_b \mathbf{T}^{-1} \tilde{\mathbf{u}}(\bar{x}_k, \cdot) = \bar{\omega}_b(\bar{x}_k)$$

for $k \in \{1, 2, 3, ..., N_1\}$ or

$$\underbrace{\begin{bmatrix} \mathbf{t}_b\mathbf{T}^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{t}_b\mathbf{T}^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{t}_b\mathbf{T}^{-1} \end{bmatrix}}_{N_1 \times M} \begin{bmatrix} \tilde{\mathbf{u}}(\bar{x}_1, \cdot) \\ \tilde{\mathbf{u}}(\bar{x}_2, \cdot) \\ \vdots \\ \tilde{\mathbf{u}}(\bar{x}_{N_1}, \cdot) \end{bmatrix} = \begin{bmatrix} \bar{\omega}_b(\bar{x}_1) \\ \bar{\omega}_b(\bar{x}_2) \\ \vdots \\ \bar{\omega}_b(\bar{x}_{N_1}) \end{bmatrix},$$

which we denote it by $\mathbf{T}_b \tilde{\mathbf{u}} = \bar{\boldsymbol{\omega}}_b$ or $\mathbf{T}_b \mathbf{P}^{-1} \mathbf{u} = \bar{\boldsymbol{\omega}}_b$, where $\mathbf{T}^{-1}$ has size $N_2 \times N_2$ and $\mathbf{t}_b = [1, -1, 1, -1, ..., (-1)^{N_2-1}]$.

- For the top boundary condition, by letting $u(\bar{x}, 1) = \bar{\omega}_u(\bar{x})$, we have

$$u(\bar{x}_k, 1) = \sum_{n=0}^{N_2-1} c_n T_n(1) = \mathbf{t}_u \mathbf{c} = \mathbf{t}_u \mathbf{T}^{-1} \tilde{\mathbf{u}}(\bar{x}_k, \cdot) = \bar{\omega}_u(\bar{x}_k)$$

for $k \in \{1, 2, 3, ..., N_1\}$ or

$$\underbrace{\begin{bmatrix} \mathbf{t}_u\mathbf{T}^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{t}_u\mathbf{T}^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{t}_u\mathbf{T}^{-1} \end{bmatrix}}_{N_1 \times M} \begin{bmatrix} \tilde{\mathbf{u}}(\bar{x}_1, \cdot) \\ \tilde{\mathbf{u}}(\bar{x}_2, \cdot) \\ \vdots \\ \tilde{\mathbf{u}}(\bar{x}_{N_1}, \cdot) \end{bmatrix} = \begin{bmatrix} \bar{\omega}_u(\bar{x}_1) \\ \bar{\omega}_u(\bar{x}_2) \\ \vdots \\ \bar{\omega}_u(\bar{x}_{N_1}) \end{bmatrix},$$

which we denote it by $\mathbf{T}_u \tilde{\mathbf{u}} = \bar{\boldsymbol{\omega}}_u$ or $\mathbf{T}_u \mathbf{P}^{-1} \mathbf{u} = \bar{\boldsymbol{\omega}}_u$, where $\mathbf{T}^{-1}$ has size $N_2 \times N_2$ and $\mathbf{t}_u = [1, 1, 1, ..., (1)^{N_2-1}]$.

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Thus, all the boundary conditions can be represented in the matrix forms as follows: $\mathbf{T}_l\mathbf{u} = \bar{\boldsymbol{\omega}}_l$, $\mathbf{T}_r\mathbf{u} = \bar{\boldsymbol{\omega}}_r$, $\mathbf{T}_b\mathbf{P}^{-1}\mathbf{u} = \bar{\boldsymbol{\omega}}_b$ and $\mathbf{T}_u\mathbf{P}^{-1}\mathbf{u} = \bar{\boldsymbol{\omega}}_u$.

Step 6. We obtain the linear system from (16) and boundary conditions in Step 5, i.e.,

$$\left[\begin{array}{c|cccc} \mathbf{K} & \mathbf{X\Phi}_y & \mathbf{\Phi}_y & \mathbf{Y\Phi}_x & \mathbf{\Phi}_x \\ \hline \mathbf{T}_l & 0 & 0 & \cdots & 0 \\ \mathbf{T}_r & 0 & 0 & \cdots & 0 \\ \mathbf{T}_b\mathbf{P}^{-1} & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_u\mathbf{P}^{-1} & 0 & 0 & \cdots & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{g}_0 \\ \mathbf{g}_1 \end{array}\right] = \left[\begin{array}{c} \mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\beta}} \\ \hline \bar{\boldsymbol{\omega}}_l \\ \bar{\boldsymbol{\omega}}_r \\ \bar{\boldsymbol{\omega}}_b \\ \bar{\boldsymbol{\omega}}_u \end{array}\right], \tag{17}$$

which we denote the block matrix (17) by

$$\begin{bmatrix} \mathbf{K} & \mathbf{Q} \\ \mathbf{R} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{z} \\ \mathbf{w} \end{bmatrix}.$$

By using the Schur complement method [6], and assuming that $\mathbf{K}$ and $\mathbf{RK}^{-1}\mathbf{Q}$ are the nonsingular matrices, we obtain the solution

$$\mathbf{u} = \mathbf{K}^{-1}\left[\mathbf{z} - \mathbf{Q}\left(\mathbf{RK}^{-1}\mathbf{Q}\right)^{-1}\left(\mathbf{RK}^{-1}\mathbf{z} - \mathbf{w}\right)\right]. \tag{18}$$

The solution $\mathbf{u}$ in (18) is the solution on $[-1, 1] \times [-1, 1]$. The approximate solution on $[a, b] \times [c, d]$ is then obtained by using the transformations $x = \frac{1}{2}[(b-a)\bar{x} + a + b]$ and $y = \frac{1}{2}[(d-c)\bar{y} + c + d]$.

Example 4.1 Consider the boundary value problem

$$\begin{aligned} & x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = -\pi^2(x+y)\cos(\pi x)\cos(\pi y), && (x, y) \in \Omega, \\ & u(x, y) = 0, && (x, y) \in \partial\Omega, \end{aligned}$$

where $\Omega = [\frac{1}{2}, \frac{3}{2}] \times [\frac{1}{2}, \frac{3}{2}]$. The exact solution is $u^*(x, y) = \cos(\pi x)\cos(\pi y)$. By using the transformations $\bar{x} = 2x - 2$ and $\bar{y} = 2y - 2$, the problem is transformed to

$$\begin{aligned} & 4\bar{\alpha}_1(\bar{x})\frac{\partial^2 u}{\partial \bar{x}^2} + 4\bar{\alpha}_2(\bar{y})\frac{\partial^2 u}{\partial \bar{y}^2} = \bar{\beta}(\bar{x}, \bar{y}), && (\bar{x}, \bar{y}) \in \bar{\Omega}, \\ & u(\bar{x}, \bar{y}) = 0, && (\bar{x}, \bar{y}) \in \partial\bar{\Omega}, \end{aligned}$$

where $\bar{\alpha}_1(\bar{x}) = \frac{1}{2}(\bar{x}+2)$, $\bar{\alpha}_2(\bar{y}) = \frac{1}{2}(\bar{y}+2)$ and $\bar{\beta}(\bar{x}, \bar{y}) = -\pi^2[(\frac{\bar{x}+2}{2})+(\frac{\bar{y}+2}{2})]\cos[\pi(\frac{\bar{x}+2}{2})]\cos[\pi(\frac{\bar{y}+2}{2})]$. Integrating the PDE twice with respect to x and twice with respect to y and use integration by parts, the resulting equation can be written in the matrix form as $\mathbf{Ku} + \mathbf{X\Phi}_y\mathbf{f}_0 + \mathbf{\Phi}_y\mathbf{f}_1 + \mathbf{Y\Phi}_x\mathbf{g}_0 + \mathbf{\Phi}_x\mathbf{g}_1 = \mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\beta}}$, where $\mathbf{K} = 4\mathbf{A}_y^2(\bar{\boldsymbol{\alpha}}_1 - 2\mathbf{A}_x\bar{\boldsymbol{\alpha}}_{1,\bar{x}}) + 4\mathbf{A}_x^2(\bar{\boldsymbol{\alpha}}_2 - 2\mathbf{A}_y\bar{\boldsymbol{\alpha}}_{2,\bar{y}})$. From the boundary conditions, we have $\mathbf{T}_l\mathbf{u} = \mathbf{0}$, $\mathbf{T}_r\mathbf{u} = \mathbf{0}$, $\mathbf{T}_b\mathbf{P}^{-1}\mathbf{u} = \mathbf{0}$ and $\mathbf{T}_u\mathbf{P}^{-1}\mathbf{u} = \mathbf{0}$. Thus, we can construct the linear system

$$\left[\begin{array}{c|cccc} \mathbf{K} & \mathbf{X\Phi}_y & \mathbf{\Phi}_y & \mathbf{Y\Phi}_x & \mathbf{\Phi}_x \\ \hline \mathbf{T}_l & 0 & 0 & \cdots & 0 \\ \mathbf{T}_r & 0 & 0 & \cdots & 0 \\ \mathbf{T}_b\mathbf{P}^{-1} & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_u\mathbf{P}^{-1} & 0 & 0 & \cdots & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{g}_0 \\ \mathbf{g}_1 \end{array}\right] = \left[\begin{array}{c} \mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\beta}} \\ \hline \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{array}\right].$$

The average absolute error for $N_1 = N_2 = N$ is shown in Table 4.

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N	FDM	FIM(TPZ)	FIM(SIM)	FIM(CBS)
6	1.9867×10^{-2}	1.4999×10^{-2}	2.2911×10^{-3}	4.6171×10^{-4}
8	9.0409×10^{-3}	8.6390×10^{-3}	2.7489×10^{-3}	5.1774×10^{-6}
10	5.1341×10^{-3}	5.5835×10^{-3}	3.4172×10^{-4}	3.5373×10^{-8}
12	3.3016×10^{-3}	3.8971×10^{-3}	2.6421×10^{-4}	1.6518×10^{-10}

Table 4: Average absolute error for Example 4.1

Example 4.2 Consider the boundary value problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xe^y, \quad 0 < x < 2, \;\; 0 < y < 1, \\
&u(0,y) = 0, \;\; u(2,y) = 2e^y, \quad 0 \le y \le 1, \\
&u(x,0) = x, \;\; u(x,1) = ex, \quad 0 < x < 2.
\end{aligned}$$

The exact solution is $u^*(x,y) = xe^y$. By using the transformations $\bar{x} = x - 1$ and $\bar{y} = 2y - 1$, the problem is transformed to

$$\begin{aligned}
&\frac{\partial^2 u}{\partial \bar{x}^2} + 4\frac{\partial^2 u}{\partial \bar{y}^2} = \bar{\beta}(\bar{x},\bar{y}), \quad -1 < \bar{x} < 1, \;\; -1 < \bar{y} < 1, \\
&u(-1,\bar{y}) = \bar{\omega}_l(\bar{y}), \;\; u(1,\bar{y}) = \bar{\omega}_r(\bar{y}), \quad -1 \le \bar{y} \le 1, \\
&u(\bar{x},-1) = \bar{\omega}_b(\bar{x}), \;\; u(\bar{x},1) = \bar{\omega}_u(\bar{x}), \quad -1 < \bar{x} < 1,
\end{aligned}$$

where $\bar{\omega}_l(\bar{y}) = 0$, $\bar{\omega}_r(\bar{y}) = 2e^{\frac{1}{2}(\bar{y}+1)}$, $\bar{\omega}_b(\bar{x}) = \bar{x} + 1$, $\bar{\omega}_u(\bar{x}) = e(\bar{x}+1)$ and $\bar{\beta}(\bar{x},\bar{y}) = (\bar{x}+1)e^{\frac{1}{2}(\bar{y}+1)}$. Integrating the PDE twice with respect to x and twice with respect to y, the resulting equation can be written in the matrix form as $\mathbf{Ku} + \mathbf{X\Phi}_y\mathbf{f}_0 + \mathbf{\Phi}_y\mathbf{f}_1 + \mathbf{Y\Phi}_x\mathbf{g}_0 + \mathbf{\Phi}_x\mathbf{g}_1 = \mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\beta}}$, where $\mathbf{K} = \mathbf{A}_y^2 + 4\mathbf{A}_x^2$. From the boundary conditions, they can be represented in the matrix forms as $\mathbf{T}_l\mathbf{u} = \bar{\boldsymbol{\omega}}_l$, $\mathbf{T}_r\mathbf{u} = \bar{\boldsymbol{\omega}}_r$, $\mathbf{T}_b\mathbf{P}^{-1}\mathbf{u} = \bar{\boldsymbol{\omega}}_b$ and $\mathbf{T}_u\mathbf{P}^{-1}\mathbf{u} = \bar{\boldsymbol{\omega}}_u$. Thus, we can construct the linear system in the matrix form as

$$\left[\begin{array}{c|cccc}
\mathbf{K} & \mathbf{X\Phi}_y & \mathbf{\Phi}_y & \mathbf{Y\Phi}_x & \mathbf{\Phi}_x \\
\hline
\mathbf{T}_l & 0 & 0 & \cdots & 0 \\
\mathbf{T}_r & 0 & 0 & \cdots & 0 \\
\mathbf{T}_b\mathbf{P}^{-1} & \vdots & \vdots & \ddots & \vdots \\
\mathbf{T}_u\mathbf{P}^{-1} & 0 & 0 & \cdots & 0
\end{array}\right]
\left[\begin{array}{c}
\mathbf{u} \\ \hline \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{g}_0 \\ \mathbf{g}_1
\end{array}\right]
=
\left[\begin{array}{c}
\mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\beta}} \\ \hline \bar{\boldsymbol{\omega}}_l \\ \bar{\boldsymbol{\omega}}_r \\ \bar{\boldsymbol{\omega}}_b \\ \bar{\boldsymbol{\omega}}_u
\end{array}\right].$$

The average relative error for $N_1 = N_2 = N$ is shown in Table 5.

Example 4.3 Consider the following boundary value problem

$$\begin{aligned}
&x\frac{\partial^2 u}{\partial x^2} - (\sin^2 x)\frac{\partial^2 u}{\partial y^2} + (\cos^2 x)\frac{\partial^2 u}{\partial x \partial y} = \cos(2x), \;\; 0 < x < 1, \;\; 0 < y < 2, \\
&u(0,y) = \frac{y^2}{2}, \;\; u(1,y) = \frac{y^2}{2} + y + 1, \quad 0 < y < 2, \\
&u(x,0) = x, \;\; u(x,2) = 3x + 2, \quad 0 \le x \le 1.
\end{aligned}$$

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N	FDM	FIM(TPZ)	FIM(SIM)	FIM(CBS)
6	2.8602×10^{-4}	3.3388×10^{-4}	4.4469×10^{-4}	5.3868×10^{-7}
8	1.3862×10^{-4}	1.8631×10^{-4}	6.7100×10^{-5}	6.4465×10^{-10}
10	8.1203×10^{-5}	1.1861×10^{-4}	7.2075×10^{-6}	6.2452×10^{-13}
12	5.3180×10^{-5}	8.2024×10^{-5}	7.2539×10^{-6}	8.8210×10^{-13}

Table 5: Average relative error for Example 4.2

The exact solution is $u^*(x,y) = x + xy + \frac{y^2}{2}$. By using the transformations $\bar{x} = 2x - 1$ and $\bar{y} = y - 1$, the problem is transformed to

$$\begin{aligned} 4\bar{\alpha}_1 \frac{\partial^2 u}{\partial \bar{x}^2} - \bar{\alpha}_2 \frac{\partial^2 u}{\partial \bar{y}^2} + 2\bar{\alpha}_3 \frac{\partial^2 u}{\partial \bar{x} \partial \bar{y}} &= \bar{\beta}(\bar{x}, \bar{y}), \quad (\bar{x}, \bar{y}) \in \bar{\Omega}, \\ u(-1, \bar{y}) = \bar{\omega}_l(\bar{y}), \quad u(1, \bar{y}) &= \bar{\omega}_r(\bar{y}), \quad -1 < \bar{y} < 1, \\ u(\bar{x}, -1) = \bar{\omega}_b(\bar{x}), \quad u(\bar{x}, 1) &= \bar{\omega}_u(\bar{x}), \quad -1 \le \bar{x} \le 1, \end{aligned}$$

where $\bar{\alpha}_1(\bar{x}) = \frac{\bar{x}+1}{2}$, $\bar{\alpha}_2(\bar{x}) = \sin^2(\frac{\bar{x}+1}{2})$, $\bar{\alpha}_3(\bar{x}) = \cos^2(\frac{\bar{x}+1}{2})$, $\bar{\omega}_l(\bar{y}) = \frac{(\bar{y}+1)^2}{2}$, $\bar{\omega}_r(\bar{y}) = \frac{(\bar{y}+1)^2}{2} + \bar{y} + 2$, $\bar{\omega}_b(\bar{x}) = \frac{\bar{x}+1}{2}$, $\bar{\omega}_u(\bar{x}) = \frac{3\bar{x}+7}{2}$ and $\bar{\beta}(\bar{x}, \bar{y}) = \cos(\bar{x}+1)$. Integrating the PDE twice with respect to x and twice with respect to y, and use integration by parts, the resulting equation can be written in the matrix form as $\mathbf{K}\mathbf{u} + \mathbf{X}\mathbf{\Phi}_y\mathbf{f}_0 + \mathbf{\Phi}_y\mathbf{f}_1 + \mathbf{Y}\mathbf{\Phi}_x\mathbf{g}_0 + \mathbf{\Phi}_x\mathbf{g}_1 = \mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\beta}}$, where $\mathbf{K} = 4\mathbf{A}_y^2(\bar{\boldsymbol{\alpha}}_1 - 2\mathbf{A}_x\bar{\boldsymbol{\alpha}}_{1,\bar{x}}) - \mathbf{A}_x^2\bar{\boldsymbol{\alpha}}_2 + 2\mathbf{A}_x\mathbf{A}_y(\bar{\boldsymbol{\alpha}}_3 + \mathbf{A}_x\bar{\boldsymbol{\alpha}}_{3,\bar{x}})$. From the boundary conditions, we have $\mathbf{T}_l\mathbf{u} = \bar{\boldsymbol{\omega}}_l$, $\mathbf{T}_r\mathbf{u} = \bar{\boldsymbol{\omega}}_r$, $\mathbf{T}_b\mathbf{P}^{-1}\mathbf{u} = \bar{\boldsymbol{\omega}}_b$ and $\mathbf{T}_u\mathbf{P}^{-1}\mathbf{u} = \bar{\boldsymbol{\omega}}_u$. Thus, we can construct the linear system in the matrix form as

$$\left[\begin{array}{c|cccc} \mathbf{K} & \mathbf{X}\mathbf{\Phi}_y & \mathbf{\Phi}_y & \mathbf{Y}\mathbf{\Phi}_x & \mathbf{\Phi}_x \\ \hline \mathbf{T}_l & 0 & 0 & \cdots & 0 \\ \mathbf{T}_r & 0 & 0 & \cdots & 0 \\ \mathbf{T}_b\mathbf{P}^{-1} & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_u\mathbf{P}^{-1} & 0 & 0 & \cdots & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{g}_0 \\ \mathbf{g}_1 \end{array}\right] = \left[\begin{array}{c} \mathbf{A}_x^2\mathbf{A}_y^2\bar{\boldsymbol{\beta}} \\ \hline \bar{\boldsymbol{\omega}}_l \\ \bar{\boldsymbol{\omega}}_r \\ \bar{\boldsymbol{\omega}}_b \\ \bar{\boldsymbol{\omega}}_u \end{array}\right].$$

The average relative error for $N_1 = N_2 = N$ is shown in Table 6.

N	FDM	FIM(TPZ)	FIM(SIM)	FIM(CBS)
6	5.9331×10^{-2}	1.8396×10^{-3}	5.5449×10^{-3}	3.0205×10^{-6}
8	3.7332×10^{-2}	7.5380×10^{-4}	1.0807×10^{-3}	1.5466×10^{-7}
10	1.4812×10^{-3}	7.0405×10^{-4}	7.4709×10^{-4}	8.7490×10^{-11}
12	7.7338×10^{-3}	4.4175×10^{-4}	2.5193×10^{-4}	4.5580×10^{-12}

Table 6: Average relative error for Example 4.3

Figure 3 shows the graphs of the numerical solutions of Examples 4.1-4.3 with 20-node discretization along x and y axes. The numerical solutions $\mathbf{u}$ of Examples 4.1-4.3 by using our modified FIM compared with their analytical solutions are shown in Figure 4.

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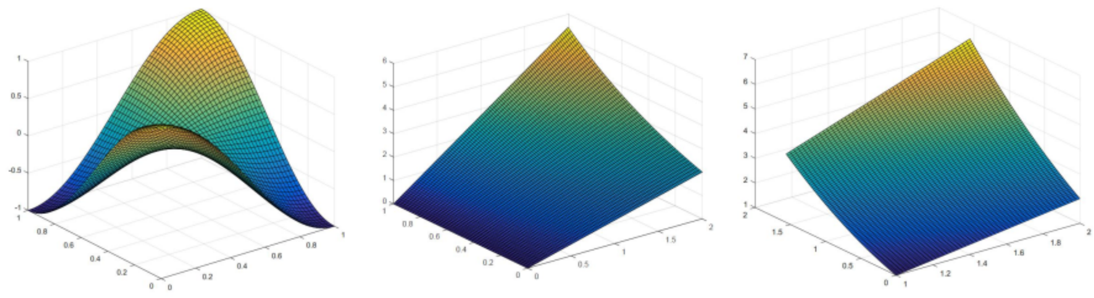
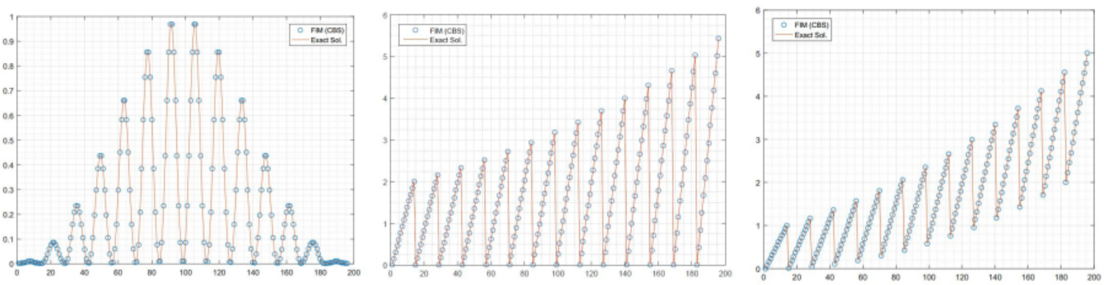


(a) Example 4.1 (b) Example 4.2 (c) Example 4.3

Figure 3: The solution surfaces of Examples 4.1-4.3



(a) Example 4.1 (b) Example 4.2 (c) Example 4.3

Figure 4: The solution at each node in Examples 4.1-4.3

5 Conclusion and Discussion

In this paper, we modify the traditional FIM. Instead of using trapezoidal or Simpson's rules as the numerical quadratures, we use the Chebyshev polynomial and its properties to construct the integral matrices of any order. By these integral matrices, we can devise procedures for solving linear ODEs and PDEs of any order. Several numerical examples show that our modified FIM significantly improves those traditional FIM in terms of accuracy with a comparatively small amount of grid points. One may see that when we compare our modified FIM with the FDM, under the same number of nodes, our modified FIM gives higher accuracy. This method engages in using larger matrix dimension to solve the systems of the linear equations. However, in order to obtain the same average absolute error, one can see from Table 7 that the FDM needs a lot more nodes. Because of this, the matrix dimension of linear system of FDM is much larger than our modified FIM.

In our future work, we will apply our modified FIM to find numerical solutions for nonlinear problems, time-dependent problems and fractional derivative problems.

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Examples	Average Relative Error	Dimension of Matrix Involved	
		FIM(CBS)	FDM
3.1	5.0×10^{-6}	9	62
3.2	5.0×10^{-3}	9	93
3.3	5.0×10^{-6}	11	36
4.1	5.0×10^{-4}	60	676
4.2	5.0×10^{-6}	52	1156
4.3	5.0×10^{-5}	52	2916

Table 7: Dimension of matrix involved when considering the same accuracy

Acknowledgement

The authors would like to thank the anonymous referees for their valuable comments and suggestions to improve the manuscript.

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