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Stable Biorthogonal Multiresolution in 3D[1]

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Abstract: Multiresolution transforms are powerful tools in video processing applications because of its flexibility in representing nonstationary signals. For a proper adaptation to the singularities, it is crucial to develop nonlinear schemes. In these applications where some coefficients are modified or discared we need to have some stability properties. In this paper, three-dimensional biorthogonal multiresolution algorithms that ensure this stability are introduced. A prescribed accuracy in various norms is ensured by these strategies. Explicit error bounds are presented.

Keywords: Stability, biorthogonal, multiresolution, non separable, video compression, quantization

Mathematics Subject Classification: 41A05,41A15

1 Introduction

Multiresolution representations are one of the most efficient tools for video compression. A particular one is the discrete multiresolution framework of Harten. It was developed to use nonlinear reconstruction processes. In image examples we can see the nonlinear process allows a better adapted treatment of edges, in the sense that they do not generate so many large detail coefficients as in the standard wavelet transforms [2], [4], [6].

In the multiresolution transforms [9] a discrete sequence $\bar{f}^L$ which represent sampling of weighted-averages of a function $f(x)$ at the finest resolution level L is encoded to produce a multi-scale representation of its information contents $(\bar{f}^0, e^1, e^2, \ldots, e^L)$, where the $\bar{f}^0$ corresponds to the sampling at the coarsest resolution level and each sequence e^k represents the intermediate details which are necessary to recover $\bar{f}^k$ from $\bar{f}^{k-1}$. This representation of the signal is well

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adapted to data compression procedures. The simplest one is obtained by setting to zero all scale coefficients which fall below a prescribed tolerance. Let us denote

$$\hat{e}^k_{i,j,l} = \mathbf{tr}(e^k_{i,j,l};\epsilon_k) = \begin{cases} 0 & |e^k_{i,j,l}| \leq \epsilon_k \\ e^k_{i,j,l} & \text{otherwise} \end{cases} \quad (1)$$

and refer to this operation as *truncation*. This type of data compression is used primarily to reduce the "dimensionality" of the data. A different strategy, which is used to reduce the digital representation of the data is *quantization*, which can be modeled by

$$\hat{e}^k_{i,j,l} = \mathbf{qu}(e^k_{i,j,l};\epsilon_k) = 2\epsilon_k \cdot \ \text{round}\left[\frac{e^k_{i,j,l}}{2\epsilon_k}\right], \quad (2)$$

where round $[\cdot]$ denotes the integer obtained by rounding. Observe that if $|e^k_{i,j,l}| < \epsilon_k \Rightarrow \mathbf{qu}(e^k_{i,j,l};\epsilon_k) = 0$ and that in both cases

$$|e^k_{i,j,l} - \hat{e}^k_{i,j,l}| \leq \epsilon_k.$$

Thus, the multi-scale representation is processed (truncation and/or quantization) and the final result of this step is a modified multi-scale representation $(\hat{f}^0, \hat{e}^1, \hat{e}^2, \ldots, \hat{e}^L)$ which is *close* to the original one, i.e. such that (in some norm)

$$||\hat{f}^0 - \bar{f}^0|| \leq \epsilon_0 \qquad ||\hat{e}^k - e^k|| \leq \epsilon_k \quad 1 \leq k \leq L,$$

where the truncation parameters $\epsilon_0, \epsilon_1, \ldots, \epsilon_L$ are chosen according to some criteria specified by the user.

After decoding the processed representation, we obtain a discrete set $\hat{f}^L$ which is expected to be *close* to the original discrete set $\bar{f}^L$. In order for this to be true, some form of stability is needed, i.e. we must require that

$$||\hat{f}^L - \bar{f}^L|| \leq \sigma(\epsilon_0, \epsilon_1, \ldots, \epsilon_L)$$

where $\sigma(\cdot, \ldots, \cdot)$ satisfies

$$\lim_{\epsilon_l \to 0,\ 0 \leq l \leq L} \sigma(\epsilon_0, \epsilon_1, \ldots, \epsilon_L) = 0.$$

The stability analysis for linear prediction processes, as wavelet basis decompositions, can be carried out using tools coming from wavelet theory, subdivision schemes and functional analysis, however none of these techniques is applicable in general when the prediction process is nonlinear [9].

In the nonlinear case, stability can be ensured by modifying the encoding algorithm. The idea of a modified-encoding to deal with nonlinear multiresolution schemes is due to Harten in the one dimensional case [5], [8]. The goal of a modified-encoding procedure is to keep track of the accumulation error in processing the values in the multi-scale representation.

In 2D, for a better adaptation to the edges we should consider the non separable approach. We introduce a modified encoding for any reconstruction type. The multivariate context of tensor product emerges only as a particular case.

The aim of this paper is to present biorthogonal three-dimensional multiresolution algorithms that ensure stability in the case of nonlinear prediction processes [3].

Summing up, using a pure 3D Harten's multiresolution transform has some advantages. The first one is its flexibility to use nonlinear techniques (good adaptation to singularities). The second one is its simplicity in both analysis: theoretical (stability) and numerical (fast algorithms) aspects.

The paper is organized as follows: We present the cell-average (3D) multiresolutions in 2. The error-control algorithms are discussed in 3. We give stability results in 4.

2 Harten's framework

Harten's framework is based on two fundamental tools: discretization $\mathcal{D}_k$ and reconstruction $\mathcal{R}_k$. The discretization operator obtains discrete information from a (non-discrete) signal ($f \in \mathcal{F}$) at a particular resolution level k. The reconstruction operator, on the other hand, produces an approximation to a signal from its discrete values. This reconstruction can be nonlinear, and then better adapted to the considered problem.

Using these two operators we can connect linear vectors spaces, V^k, that represent in some way the different resolution levels (k increasing implies more resolution), i.e.,

$$\begin{aligned} D_k^{k-1} &: V^k \to V^{k-1}, \quad \textit{decimation}, \\ P_{k-1}^k &: V^{k-1} \to V^k, \quad \textit{prediction}. \end{aligned}$$

2.1 Cell Average MR analysis in 3D

Let $X^k = \{x_i^k, y_j^k, z_l^k\}_{i,j,l=0}^{J_k}$, $J_k = 2^k J_0$, J_0 some integer, $x_{2i}^k = x_i^{k-1}$, $y_{2j}^k = y_j^{k-1}$ and $z_{2l}^k = z_l^{k-1}$.

We define the discretization operator

$$\mathcal{D}_k : L^1([0,1]\times[0,1]\times[0,1]) \longrightarrow V^k, \tag{3}$$

$$\bar{f}_{i,j,m}^k = (\mathcal{D}_k f)_{i,j,m} = \frac{1}{h_k^3}\int_{x_{i-1}^k}^{x_i^k}\int_{y_{j-1}^k}^{y_j^k}\int_{z_{m-1}^k}^{z_m^k} f(x,y,z)dzdydx, \quad 1 \le i,j,m \le J_k; \tag{4}$$

where $L^1([0,1]\times[0,1]\times[0,1])$ is the space of absolutely integrable functions in $[0,1]\times[0,1]\times[0,1]$, and h_k is the uniform spacing of the grid points.

This analysis turns out to be appropriate for data compression of discontinuous, piecewise smooth signals.

It is sufficient to consider weighted averages $\bar{f}_{i,j,m}^k$ for $1 \le i,j,m \le J_k$ since these contain information on f over $[0,1]\times[0,1]\times[0,1]$. Thus, V^k is the space of sequences with $J_k \times J_k \times J_k$ components.

Moreover

$$\begin{aligned} \bar{f}_{i,j,m}^{k-1} &= (D_k^{k-1}\bar{f}^k)_{i,j,m} \\ &= \frac{1}{8}(\bar{f}_{2i-1,2j-1,2m-1}^k + \bar{f}_{2i-1,2j,2m-1}^k + \bar{f}_{2i,2j-1,2m-1}^k + \bar{f}_{2i,2j,2m-1}^k \\ &\quad + \bar{f}_{2i-1,2j-1,2m}^k + \bar{f}_{2i-1,2j,2m}^k + \bar{f}_{2i,2j-1,2m}^k + \bar{f}_{2i,2j,2m}^k) \end{aligned}$$

$i,j,m = 1,2,\ldots,J_{k-1}$.

On the other hand, since

$$\begin{aligned} 0 = (D_k^{k-1}e^k)_{i,j,m} &= (e_{2i-1,2j-1,2m-1}^k + e_{2i-1,2j,2m-1}^k + e_{2i,2j-1,2m-1}^k + e_{2i,2j,2m-1}^k \\ &\quad + e_{2i-1,2j-1,2m}^k + e_{2i-1,2j,2m}^k + e_{2i,2j-1,2m}^k + e_{2i,2j,2m}^k)/8 \end{aligned}$$

we do not need to keep $e_{2i,2j,2m}^k$.

A reconstruction operator for this discretization is any operator $\mathcal{R}_k$ satisfying

$$\mathcal{R}_k : V_k \longrightarrow L^1([0,1]\times[0,1]\times[0,1]),$$

$$(\mathcal{D}_k\mathcal{R}_k\bar{f}^k)_{i,j,m} = \frac{1}{h_k^3}\int_{x_{i-1}^k}^{x_i^k}\int_{y_{j-1}^k}^{y_j^k}\int_{z_{m-1}^k}^{z_m^k} (\mathcal{R}_k\bar{f}^k)(x,y,z)dzdydx = \bar{f}_{i,j,m}^k.$$

That is, $\mathcal{R}_k\bar{f}^k(x,y)$ has to be a function in $L^1([0,1]\times[0,1]\times[0,1])$ whose mean value on the (i,j,m)-th cell coincides with $\bar{f}_{i,j,m}^k$, $\forall i,j,m$. Finally, $P_{k-1}^k := \mathcal{D}_k\mathcal{R}_{k-1}$.

The multiresolution transform and its inverse are now

Algorithm 1 $M\bar{f}^L$ *(Encoding)*

for $k = L, \dots, 1$
  for $i, j, m = 1, \dots, J_{k-1}$
    $\bar{f}^{k-1}_{i,j,m} = \frac{1}{8}(\bar{f}^k_{2i-1,2j-1,2m-1} + \bar{f}^k_{2i-1,2j,2m-1} + \bar{f}^k_{2i,2j-1,2m-1} + \bar{f}^k_{2i,2j,2m-1} + \bar{f}^k_{2i-1,2j-1,2m} + \bar{f}^k_{2i-1,2j,2m} + \bar{f}^k_{2i,2j-1,2m} + \bar{f}^k_{2i,2j,2m})$
  end
  for $i, j, m = 1, \dots, J_{k-1}$
    $e^k_{2i-1,2j-1,2m-1} = \bar{f}^k_{2i-1,2j-1,2m-1} - (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j-1,2m-1}$
    $e^k_{2i-1,2j,2m-1} = \bar{f}^k_{2i-1,2j,2m-1} - (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j,2m-1}$
    $e^k_{2i,2j-1,2m-1} = \bar{f}^k_{2i,2j-1,2m-1} - (P^k_{k-1}\bar{f}^{k-1})_{2i,2j-1,2m-1}$
    $e^k_{2i,2j,2m-1} = \bar{f}^k_{2i,2j,2m-1} - (P^k_{k-1}\bar{f}^{k-1})_{2i,2j,2m-1}$
    $e^k_{2i-1,2j-1,2m} = \bar{f}^k_{2i-1,2j-1,2m} - (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j-1,2m}$
    $e^k_{2i-1,2j,2m} = \bar{f}^k_{2i-1,2j,2m} - (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j,2m}$
    $e^k_{2i,2j-1,2m} = \bar{f}^k_{2i,2j-1,2m} - (P^k_{k-1}\bar{f}^{k-1})_{2i,2j-1,2m}$
  end
end

$M\bar{f}^L = \{\bar{f}^0, e^1, \dots, e^L\}$

Algorithm 2 $\bar{f}^L = M^{-1}M\bar{f}^L$ *(Decoding)*

for $k = 1, \dots, L$
  for $i, j, m = 1, \dots, J_{k-1}$
    $\bar{f}^k_{2i-1,2j-1,2m-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j-1,2m-1} + e^k_{2i-1,2j-1,2m-1}$
    $\bar{f}^k_{2i-1,2j,2m-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j,2m-1} + e^k_{2i-1,2j,2m-1}$
    $\bar{f}^k_{2i,2j-1,2m-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i,2j-1,2m-1} + e^k_{2i,2j-1,2m-1}$
    $\bar{f}^k_{2i,2j,2m-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i,2j,2m-1} + e^k_{2i,2j,2m-1}$
    $\bar{f}^k_{2i-1,2j-1,2m} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j-1,2m} + e^k_{2i-1,2j-1,2m}$
    $\bar{f}^k_{2i-1,2j,2m} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j,2m} + e^k_{2i-1,2j,2m}$
    $\bar{f}^k_{2i,2j-1,2m} = (P^k_{k-1}\bar{f}^{k-1})_{2i,2j-1,2m} + e^k_{2i,2j-1,2m}$
    $\bar{f}^k_{2i,2j,2m} = 8\bar{f}^{k-1}_{i,j,m} - \bar{f}^k_{2i-1,2j-1,2m-1} - \bar{f}^k_{2i-1,2j,2m-1} - \bar{f}^k_{2i,2j-1,2m-1} - \bar{f}^k_{2i,2j,2m-1} - \bar{f}^k_{2i-1,2j-1,2m} - \bar{f}^k_{2i-1,2j,2m} - \bar{f}^k_{2i,2j-1,2m}$
  end
end

3 Multiresolution based compression transformations using error control

By applying the inverse multiresolution transform to the compressed representation, we obtain $\hat{f}^L = M^{-1}\{\hat{f}^0, \hat{e}^1, \dots, \hat{e}^L\}$, an approximation to the original signal $\bar{f}^L$. We expect the information contents of $\hat{f}^L$ to be very close to those of the original signal $\bar{f}^L$, and in order for this to be true, the stability of the multiresolution scheme with respect to perturbations is essential. Studying the effect of using $\hat{e}^k_{i,j,l}$ (1)-(2) instead of $e^k_{i,j,l}$ in the input of M^{-1} is equivalent to studying the effect of a perturbation in the scale coefficients in the outcome of the inverse multiresolution transform.

Given a discrete sequence $\bar{f}^L$ and a tolerance level ϵ for accuracy, our task is to come up with a compressed representation

$$\{\hat{f}^0, \hat{e}^1, \ldots, \hat{e}^L\} \tag{5}$$

such that if $\hat{f}^L = M^{-1}\{\hat{f}^0, \hat{e}^1, \ldots, \hat{e}^L\}$, we have

$$\| \bar{f}^L - \hat{f}^L \| \leq C\epsilon \tag{6}$$

for an appropriate norm.

As observed by Harten [8], one possible way to accomplish this goal is to modify the encoding procedure in such a way that the modification allows us to keep track of the cumulative error and process (truncation and quantization) accordingly.

In what follows we present a three-dimensional extension of the one dimensional algorithms in [8], [5]. For a proper adaptation to the singularities, in some cases, it is crucial to develop nonlinear methods which are not based on tensor product. Thus, one needs to control the stability of these representations.

Given a tolerance level ϵ, the outcome of the modified encoding procedure is a compressed representation (5) satisfying (6). A modified encoding procedure is designed keeping in mind the particular decoding procedure to be used.

We use the following norms:

$$\begin{aligned}
\|\bar{f}^k\|_\infty &= \sup_{i,j,l}|\bar{f}^k_{i,j,l}|, \\
\|\bar{f}^k\|_1 &= \frac{1}{J_k^3}(\sum_{i,j,l}|\bar{f}^k_{i,j,l}|), \\
\|\bar{f}^k\|_2^2 &= \frac{1}{J_k^3}(\sum_{i,j,l}|\bar{f}^k_{i,j,l}|^2),
\end{aligned}$$

where $\bar{f}^k = \{\bar{f}^k_{ijl}\}$.

We denote by **pr** the compression process (truncation and quantization).

3.1 Error-control algorithms

We denote
$\tilde{e}^k_{\cdot,\cdot,\cdot} = (\bar{f}^k_{\cdot,\cdot,\cdot} - P^k_{k-1}\hat{f}^{k-1})_{\cdot,\cdot,\cdot}) - (\bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m})$.

Algorithm 3 *Modified encoding for cell-average*

for $k = L, \ldots, 1$
  for $i, j, m = 1, \ldots, J_{k-1}$
    $\bar{f}^{k-1}_{i,j,m} = \frac{1}{8}(\bar{f}^k_{2i-1,2j-1,2m-1} + \bar{f}^k_{2i-1,2j,2m-1} + \bar{f}^k_{2i,2j-1,2m-1} + \bar{f}^k_{2i,2j,2m-1}$
      $+\bar{f}^k_{2i-1,2j-1,2m} + \bar{f}^k_{2i-1,2j,2m} + \bar{f}^k_{2i,2j-1,2m} + \bar{f}^k_{2i,2j,2m})$
  end
end
Set $\hat{f}^0 = \bar{f}^0$
for $k = 1, \ldots, L$

for $i, j, m = 1, \ldots, J_{k-1}$

$\hat{e}^k_{2i-1,2j-1,2m-1} = \mathbf{pr}(\tilde{e}^k_{2i-1,2j-1,2m-1}, \epsilon_k)$

$\hat{f}^k_{2i-1,2j-1,2m-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j-1,2m-1} + \hat{e}^k_{2i-1,2j-1,2m-1}$

$\hat{e}^k_{2i-1,2j,2m-1} = \mathbf{pr}(\tilde{e}^k_{2i-1,2j,2m-1}, \epsilon_k)$

$\hat{f}^k_{2i-1,2j,2m-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j,2m-1} + \hat{e}^k_{2i-1,2j,2m-1}$

$\hat{e}^k_{2i,2j-1,2m-1} = \mathbf{pr}(\tilde{e}^k_{2i,2j-1,2m-1}, \epsilon_k)$

$\hat{f}^k_{2i,2j-1,2m-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i,2j-1,2m-1} + \hat{e}^k_{2i,2j-1,2m-1}$

$\hat{e}^k_{2i,2j,2m-1} = \mathbf{pr}(\tilde{e}^k_{2i,2j,2m-1}, \epsilon_k)$

$\hat{f}^k_{2i,2j,2m-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i,2j,2m-1} + \hat{e}^k_{2i,2j,2m-1}$

$\hat{e}^k_{2i-1,2j-1,2m} = \mathbf{pr}(\tilde{e}^k_{2i-1,2j-1,2m}, \epsilon_k)$

$\hat{f}^k_{2i-1,2j-1,2m} = (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j-1,2m} + \hat{e}^k_{2i-1,2j-1,2m}$

$\hat{e}^k_{2i-1,2j,2m} = \mathbf{pr}(\tilde{e}^k_{2i-1,2j,2m}, \epsilon_k)$

$\hat{f}^k_{2i-1,2j,2m} = (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j,2m} + \hat{e}^k_{2i-1,2j,2m}$

$\hat{e}^k_{2i,2j-1,2m} = \mathbf{pr}(\tilde{e}^k_{2i,2j-1,2m}, \epsilon_k)$

$\hat{f}^k_{2i,2j-1,2m} = (P^k_{k-1}\hat{f}^{k-1})_{2i,2j-1,2m} + \hat{e}^k_{2i,2j-1,2m}$

$$\hat{f}^k_{2i,2j,2m} = 8\hat{f}^{k-1}_{i,j,m} - \hat{f}^k_{2i-1,2j-1,2m-1} - \hat{f}^k_{2i-1,2j,2m-1} - \hat{f}^k_{2i,2j-1,2m-1} - \hat{f}^k_{2i,2j,2m-1} - \hat{f}^k_{2i-1,2j-1,2m} - \hat{f}^k_{2i-1,2j,2m} - \hat{f}^k_{2i,2j-1,2m}$$

end

end

4 Stability and explicit error bounds

Let us denote $e^k_{i,j,m}(1) = e^k_{2i-1,2j-1,2m-1}$, $e^k_{i,j,m}(2) = e^k_{2i,2j-1,2m-1}$, $e^k_{i,j,m}(3) = e^k_{2i-1,2j,2m-1}$, $e^k_{i,j,m}(4) = e^k_{2i-1,2j-1,2m}$, $e^k_{i,j,m}(5) = e^k_{2i,2j,2m-1}$, $e^k_{i,j,m}(6) = e^k_{2i,2j-1,2m}$, $e^k_{i,j,m}(7) = e^k_{2i-1,2j,2m}$.

Proposition 1 *Given a discrete sequence $\bar{f}^L$, with the modified encoding algorithm for the cell-average framework in 3D (Algorithm 3) we obtain a multiresolution representation $M\bar{f}^L = \{\hat{f}^0, \hat{e}^1, \ldots, \hat{e}^L\}$ such that if we apply the decoding algorithm we obtain $\hat{f}^L$ satisfying:*

$$||\bar{f}^L - \hat{f}^L||_\infty \leq ||\bar{f}^0 - \hat{f}^0||_\infty + 7\sum_{k=1}^{L} |||\tilde{e}^k - \hat{e}^k|||_\infty \tag{7}$$

$$||\bar{f}^L - \hat{f}^L||_1 \leq ||\bar{f}^0 - \hat{f}^0||_1 + \frac{1}{4}\sum_{k=1}^{L} |||\tilde{e}^k - \hat{e}^k|||_1 \tag{8}$$

$$||\bar{f}^L - \hat{f}^L||_2^2 = ||\bar{f}^0 - \hat{f}^0||_2^2 + \frac{1}{8}\sum_{k=1}^{L} |||\tilde{e}^k - \hat{e}^k|||_2^2 + \frac{1}{8}\sum_{k=1}^{L} ||\tilde{E}^k - \hat{E}^k||_2^2 \tag{9}$$

where

$$
\begin{aligned}
|||\tilde{e}^k - \hat{e}^k|||_\infty &= \max(||\tilde{e}^k(1) - \hat{e}^k(1)||_\infty, ||\tilde{e}^k(2) - \hat{e}^k(2)||_\infty,\\
&\quad ||\tilde{e}^k(3) - \hat{e}^k(3)||_\infty, ||\tilde{e}^k(4) - \hat{e}^k(4)||_\infty,\\
&\quad ||\tilde{e}^k(5) - \hat{e}^k(5)||_\infty, ||\tilde{e}^k(6) - \hat{e}^k(6)||_\infty,\\
&\quad ||\tilde{e}^k(7) - \hat{e}^k(7)||_\infty),\\
|||\tilde{e}^k - \hat{e}^k|||_1 &= ||\tilde{e}^k(1) - \hat{e}^k(1)||_1 + ||\tilde{e}^k(2) - \hat{e}^k(2)||_1 +\\
&\quad ||\tilde{e}^k(3) - \hat{e}^k(3)||_1 + ||\tilde{e}^k(4) - \hat{e}^k(4)||_1 +\\
&\quad ||\tilde{e}^k(5) - \hat{e}^k(5)||_1 + ||\tilde{e}^k(6) - \hat{e}^k(6)||_1 +\\
&\quad ||\tilde{e}^k(7) - \hat{e}^k(7)||_1,\\
|||\tilde{e}^k - \hat{e}^k|||_2^2 &= ||\tilde{e}^k(1) - \hat{e}^k(1)||_2^2 + ||\tilde{e}^k(2) - \hat{e}^k(2)||_2^2 +\\
&\quad ||\tilde{e}^k(3) - \hat{e}^k(3)||_2^2 + ||\tilde{e}^k(4) - \hat{e}^k(4)||_2^2 +\\
&\quad ||\tilde{e}^k(5) - \hat{e}^k(5)||_2^2 + ||\tilde{e}^k(6) - \hat{e}^k(6)||_2^2 +\\
&\quad ||\tilde{e}^k(7) - \hat{e}^k(7)||_2^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{E}^k_{i,j,m} - \hat{E}^k_{i,j,m} &= ((\tilde{e}^k_{2i-1,2j,2m-1} - \hat{e}^k_{2i-1,2j,2m-1}) + (\tilde{e}^k_{2i-1,2j-1,2m-1} - \hat{e}^k_{2i-1,2j-1,2m-1})\\
&\quad +(\tilde{e}^k_{2i,2j-1,2m-1} - \hat{e}^k_{2i,2j-1,2m-1}) + (\tilde{e}^k_{2i,2j,2m-1} - \hat{e}^k_{2i,2j,2m-1})\\
&\quad +((\tilde{e}^k_{2i-1,2j,2m} - \hat{e}^k_{2i-1,2j,2m}) + (\tilde{e}^k_{2i-1,2j-1,2m} - \hat{e}^k_{2i-1,2j-1,2m})\\
&\quad +(\tilde{e}^k_{2i,2j-1,2m} - \hat{e}^k_{2i,2j-1,2m})).
\end{aligned}
$$

Proof

From the encoding algorithm we obtain:

$$
\begin{aligned}
\bar{f}^k_{2i-1,2j,2m-1} - \hat{f}^k_{2i-1,2j,2m-1} &= \bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m} + (\tilde{e}^k_{2i-1,2j,2m-1} - \hat{e}^k_{2i-1,2j,2m-1}) \\
\bar{f}^k_{2i,2j-1,2m-1} - \hat{f}^k_{2i,2j-1,2m-1} &= \bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m} + (\tilde{e}^k_{2i,2j-1,2m-1} - \hat{e}^k_{2i,2j-1,2m-1}) \\
\bar{f}^k_{2i-1,2j-1,2m-1} - \hat{f}^k_{2i-1,2j-1,2m-1} &= \bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m} + (\tilde{e}^k_{2i-1,2j-1,2m-1} - \hat{e}^k_{2i-1,2j-1,2m-1}) \\
\bar{f}^k_{2i,2j,2m-1} - \hat{f}^k_{2i,2j,2m-1} &= \bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m} + (\tilde{e}^k_{2i,2j,2m-1} - \hat{e}^k_{2i,2j,2m-1}) \\
\bar{f}^k_{2i-1,2j,2m} - \hat{f}^k_{2i-1,2j,2m} &= \bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m} + (\tilde{e}^k_{2i-1,2j,2m} - \hat{e}^k_{2i-1,2j,2m}) \\
\bar{f}^k_{2i,2j-1,2m} - \hat{f}^k_{2i,2j-1,2m} &= \bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m} + (\tilde{e}^k_{2i,2j-1,2m} - \hat{e}^k_{2i,2j-1,2m}) \\
\bar{f}^k_{2i-1,2j-1,2m} - \hat{f}^k_{2i-1,2j-1,2m} &= \bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m} + (\tilde{e}^k_{2i-1,2j-1,2m} - \hat{e}^k_{2i-1,2j-1,2m}) \\
\bar{f}^k_{2i,2j,2m} - \hat{f}^k_{2i,2j,2m} &= \bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m} - (\tilde{e}^k_{2i-1,2j,2m-1} - \hat{e}^k_{2i-1,2j,2m-1}) \\
& -(\tilde{e}^k_{2i,2j-1,2m-1} - \hat{e}^k_{2i,2j-1,2m-1}) \\
& -(\tilde{e}^k_{2i-1,2j-1,2m-1} - \hat{e}^k_{2i-1,2j-1,2m-1}) \\
& -(\tilde{e}^k_{2i,2j,2m-1} - \hat{e}^k_{2i,2j,2m-1}) \\
& -(\tilde{e}^k_{2i-1,2j,2m} - \hat{e}^k_{2i-1,2j,2m}) \\
& -(\tilde{e}^k_{2i,2j-1,2m} - \hat{e}^k_{2i,2j-1,2m}) \\
& -(\tilde{e}^k_{2i-1,2j-1,2m} - \hat{e}^k_{2i-1,2j-1,2m}).
\end{aligned}
$$

Then

$$
\begin{aligned}
||\bar{f}^k - \hat{f}^k||_\infty \quad \leq \quad & ||\bar{f}^{k-1} - \hat{f}^{k-1}||_\infty + 7\max(||\tilde{e}^k(1) - \hat{e}^k(1)||_\infty, \\
& ||\tilde{e}^k(2) - \hat{e}^k(2)||_\infty, ||\tilde{e}^k(3) - \hat{e}^k(3)||_\infty \\
& ||\tilde{e}^k(4) - \hat{e}^k(4)||_\infty, ||\tilde{e}^k(5) - \hat{e}^k(5)||_\infty \\
& ||\tilde{e}^k(6) - \hat{e}^k(6)||_\infty, ||\tilde{e}^k(7) - \hat{e}^k(7)||_\infty)
\end{aligned}
$$

and we obtain (7).

Taking $p = 1$,

$$
\begin{aligned}
& |\bar{f}^k_{2i,2j,2m-1} - \hat{f}^k_{2i,2j,2m-1}| + |\bar{f}^k_{2i-1,2j,2m-1} - \hat{f}^k_{2i-1,2j,2m-1}| \\
& +|\bar{f}^k_{2i-1,2j-1,2m-1} - \hat{f}^k_{2i-1,2j-1,2m-1}| + |\bar{f}^k_{2i,2j-1,2m-1} - \hat{f}^k_{2i,2j-1,2m-1}| \\
& +|\bar{f}^k_{2i,2j,2m} - \hat{f}^k_{2i,2j,2m}| + |\bar{f}^k_{2i-1,2j,2m} - \hat{f}^k_{2i-1,2j,2m}| \\
& +|\bar{f}^k_{2i-1,2j-1,2m} - \hat{f}^k_{2i-1,2j-1,2m}| + |\bar{f}^k_{2i,2j-1,2m} - \hat{f}^k_{2i,2j-1,2m}| \\
& \leq 8|\bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m}| + 2(|\tilde{e}^k_{2i-1,2j,2m-1} - \hat{e}^k_{2i-1,2j,2m-1}| + \\
& |\tilde{e}^k_{2i-1,2j-1,2m-1} - \hat{e}^k_{2i-1,2j-1,2m-1}| + |\tilde{e}^k_{2i,2j-1,2m-1} - \hat{e}^k_{2i,2j-1,2m-1}| \\
& +|\tilde{e}^k_{2i,2j,2m-1} - \hat{e}^k_{2i,2j,2m-1}| + |\tilde{e}^k_{2i-1,2j,2m} - \hat{e}^k_{2i-1,2j,2m}| + \\
& |\tilde{e}^k_{2i-1,2j-1,2m} - \hat{e}^k_{2i-1,2j-1,2m}| + |\tilde{e}^k_{2i,2j-1,2m} - \hat{e}^k_{2i,2j-1,2m}|)
\end{aligned}
$$

From the previous inequality and from the definition of $||\cdot||_1$-norm ($J_k = 2J_{k-1}$) we obtain

$$\begin{array}{rcl} ||\bar{f}^k - \hat{f}^k||_1 & \leq & ||\bar{f}^{k-1} - \hat{f}^{k-1}||_1 + \frac{1}{4}(||\tilde{e}^k(1) - \hat{e}^k(1)||_1 \\ & + & ||\tilde{e}^k(2) - \hat{e}^k(2)||_1 + ||\tilde{e}^k(3) - \hat{e}^k(3)||_1 \\ & + & ||\tilde{e}^k(4) - \hat{e}^k(4)||_1 + ||\tilde{e}^k(5) - \hat{e}^k(5)||_1 \\ & + & ||\tilde{e}^k(6) - \hat{e}^k(6)||_1 + ||\tilde{e}^k(7) - \hat{e}^k(7)||_1) \end{array}$$

which proves (8).

Finally, from

$$\begin{array}{l} (\bar{f}^k_{2i,2j,2m-1} - \hat{f}^k_{2i,2j,2m-1})^2 + (\bar{f}^k_{2i-1,2j,2m-1} - \hat{f}^k_{2i-1,2j,2m-1})^2 \\ +(\bar{f}^k_{2i-1,2j-1,2m-1} - \hat{f}^k_{2i-1,2j-1,2m-1})^2 + (\bar{f}^k_{2i,2j-1,2m-1} - \hat{f}^k_{2i,2j-1,2m-1})^2 \\ +(\bar{f}^k_{2i,2j,2m} - \hat{f}^k_{2i,2j,2m})^2 + (\bar{f}^k_{2i-1,2j,2m} - \hat{f}^k_{2i-1,2j,2m})^2 \\ (\bar{f}^k_{2i-1,2j-1,2m} - \hat{f}^k_{2i-1,2j-1,2m})^2 + (\bar{f}^k_{2i,2j-1,2m} - \hat{f}^k_{2i,2j-1,2m})^2 \\ = 8(\bar{f}^{k-1}_{i,j,m} - \hat{f}^{k-1}_{i,j,m})^2 + (\tilde{e}^k_{2i-1,2j,2m-1} - \hat{e}^k_{2i-1,2j,2m-1})^2 \\ +(\tilde{e}^k_{2i-1,2j-1,2m-1} - \hat{e}^k_{2i-1,2j-1,2m-1})^2 + (\tilde{e}^k_{2i,2j-1,2m-1} - \hat{e}^k_{2i,2j-1,2m-1})^2 \\ +(\tilde{e}^k_{2i,2j,2m-1} - \hat{e}^k_{2i,2j,2m-1})^2 + (\tilde{e}^k_{2i,2j-1,2m} - \hat{e}^k_{2i,2j-1,2m})^2 \\ +(\tilde{e}^k_{2i-1,2j-1,2m} - \hat{e}^k_{2i-1,2j-1,2m})^2 + (\tilde{e}^k_{2i,2j-1,2m} - \hat{e}^k_{2i,2j-1,2m})^2 \\ +((\tilde{e}^k_{2i-1,2j,2m-1} - \hat{e}^k_{2i-1,2j,2m-1}) + (\tilde{e}^k_{2i-1,2j-1,2m-1} - \hat{e}^k_{2i-1,2j-1,2m-1}) \\ +(\tilde{e}^k_{2i,2j-1,2m-1} - \hat{e}^k_{2i,2j-1,2m-1}) + (\tilde{e}^k_{2i,2j-1,2m-1} - \hat{e}^k_{2i,2j-1,2m-1}) \\ +(\tilde{e}^k_{2i-1,2j,2m} - \hat{e}^k_{2i-1,2j,2m}) + (\tilde{e}^k_{2i-1,2j-1,2m} - \hat{e}^k_{2i-1,2j-1,2m}) \\ +(\tilde{e}^k_{2i,2j-1,2m} - \hat{e}^k_{2i,2j-1,2m}))^2 \end{array}$$

we obtain

$$\begin{array}{rcl} ||\bar{f}^k - \hat{f}^k||_2^2 & = & ||\bar{f}^{k-1} - \hat{f}^{k-1}||_2^2 + \frac{1}{8}(||\tilde{e}^k(1) - \hat{e}^k(1)||_2^2 \\ & + & ||\tilde{e}^k(2) - \hat{e}^k(2)||_2^2 + ||\tilde{e}^k(3) - \hat{e}^k(3)||_2^2 \\ & + & ||\tilde{e}^k(4) - \hat{e}^k(4)||_2^2 + ||\tilde{e}^k(5) - \hat{e}^k(5)||_2^2 \\ & + & ||\tilde{e}^k(6) - \hat{e}^k(6)||_2^2 + ||\tilde{e}^k(7) - \hat{e}^k(7)||_2^2 \\ & + & ||\tilde{E}^k - \hat{E}^k||_2^2). \end{array}$$

And the proposition has been proved.

□

It is absolutely trivial then to prove the following corollary.

Corollary 1 *Consider the error control multiresolution scheme described in Proposition 1, and a processing strategy for the scale coefficients such that*

$$||\tilde{e}^k(l) - \hat{e}^k(l)||_p \leq \epsilon_k \quad l = 1,2,3, \quad p = \infty, 1, \textit{ or } 2 \tag{10}$$

Then we have

$$\begin{aligned} ||\bar{f}^L - \hat{f}^L||_\infty &\leq ||\bar{f}^0 - \hat{f}^0||_\infty + 7\sum_{k=1}^{L}\epsilon_k \\ ||\bar{f}^L - \hat{f}^L||_1 &\leq ||\bar{f}^0 - \hat{f}^0||_1 + \frac{7}{4}\sum_{k=1}^{L}\epsilon_k \\ ||\bar{f}^L - \hat{f}^L||_2^2 &\leq ||\bar{f}^0 - \hat{f}^0||_2^2 + 7\sum_{k=1}^{L}\epsilon_k^2 \end{aligned}$$

In particular, if we assume that $||\bar{f}^0 - \hat{f}^0|| = 0$ *and we consider*

$$\epsilon_k = \frac{\epsilon}{q^{L-k+1}} \qquad q > 1 \tag{11}$$

we obtain

$$\begin{aligned} ||\bar{f}^L - \hat{f}^L||_\infty &\leq \frac{7\epsilon}{q-1}, \\ ||\bar{f}^L - \hat{f}^L||_1 &\leq \frac{7}{4}\frac{\epsilon}{q-1}, \\ ||\bar{f}^L - \hat{f}^L||_2^2 &\leq \frac{7\epsilon^2}{q^2-1}. \end{aligned}$$

These bounds become

$$||\bar{f}^L - \hat{f}^L||_\infty \leq 7\epsilon; \quad ||\bar{f}^L - \hat{f}^L||_1 \leq \frac{7}{2}\epsilon; \quad or \quad ||\bar{f}^L - \hat{f}^L||_2 \leq \frac{7}{3}\epsilon \tag{12}$$

for $q = 2$.

Proof
Notice that,

$$\begin{aligned} ||\tilde{E}^k - \hat{E}^k||_2 &\leq ||\tilde{e}^k(1) - \hat{e}^k(1)||_2 + ||\tilde{e}^k(2) - \hat{e}^k(2)||_2 \\ &+ ||\tilde{e}^k(3) - \hat{e}^k(3)||_2 + ||\tilde{e}^k(4) - \hat{e}^k(4)||_2 \\ &+ ||\tilde{e}^k(5) - \hat{e}^k(5)||_2 + ||\tilde{e}^k(6) - \hat{e}^k(6)||_2 \\ &+ ||\tilde{e}^k(7) - \hat{e}^k(7)||_2 \end{aligned}$$

Then from (9) we obtain,

$$\begin{aligned} ||\bar{f}^L - \hat{f}^L||_2^2 &\leq ||\bar{f}^0 - \hat{f}^0||_2^2 + \frac{1}{8}\sum_{k=1}^{L}7\epsilon_k^2 + \frac{1}{8}\sum_{k=1}^{L}49\epsilon_k^2 \\ &= ||\bar{f}^0 - \hat{f}^0||_2^2 + 7\sum_{k=1}^{L}\epsilon_k^2 \end{aligned}$$

□

The results of Proposition 1 can be viewed as a-posteriori bounds on the compression error. The schemes permit to monitor the compression error at each resolution level [1]. This fact allows us to design new processing strategies that aim at reducing as much as possible the number of nonzero coefficients, at the same time, that the total compression error is kept below a specific tolerance.

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