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# The Application of Asymptotic Analysis for Developing Reliable Numerical Method for a Model Singular Perturbation Problem[1]

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*Abstract:* This paper deals with the design and implementation of some appropriate numerical methods for a model singularly perturbed differential equation. To adapt the local abrupt changes of the solution, we consider a piecewise uniform mesh instead of uniform mesh throughout the underlying interval. For designing appropriate grid, some *apriori* information about the width of boundary layer (denoted by $\delta$) is always helpful. We use asymptotic analysis to find out this information about $\delta$. Based on this information on $\delta$, we design a fitted mesh numerical method. Numerical results for this method and those corresponding to its standard analogue are presented.



*Keywords:* Singular perturbation problem; asymptotic analysis; finite difference methods.

*Mathematics Subject Classification:* 34E10, 34E15, 65L10, 65L12.

## 1 Introduction

Consider a linear singular perturbation problem (SPP)

$$P_\varepsilon := \mathcal{L}y \equiv \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x), \quad x \in (0,1), \tag{1}$$

with the boundary conditions

$$y(0) = \alpha_0, \quad y(1) = \alpha_1; \quad \alpha_0,\, \alpha_1 \in \mathbb{R}, \tag{2}$$

where $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions and satisfy $a(x) \geq a > 0$, $b(x) \geq b > 0$ and $\varepsilon$ is a very small positive parameter (known as the singular perturbation parameter). These conditions on the coefficient functions guarantee the existence of a unique solution. Furthermore, $a(x)$ is assumed to be of the same sign so as to avoid the occurrence of interior layers. Under the conditions on the coefficients functions in (1), the operator $\mathcal{L}$ satisfies

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**Lemma 1.1** (Continuous minimum principle) *Assume that* $\Pi(x)$ *is any sufficiently smooth function satisfying* $\Pi(0) \geq 0$ *and* $\Pi(1) \geq 0$. *Then* $\mathcal{L}\Pi(x) \leq 0$ *for all* $x \in (0,1)$ *implies that* $\Pi(x) \geq 0$ *for all* $x \in [0,1]$.

**Proof.** Let $x^*$ be such that $\Pi(x^*) = \min_{x\in[0,1]} \Pi(x)$ and assume that $\Pi(x^*) < 0$. Clearly $x^* \notin \{0,1\}$, therefore $y'(x^*) = 0$ and $y''(x^*) \geq 0$. Thus,

$$\begin{aligned} \mathcal{L}\Pi(x^*) &= \varepsilon y''(x^*) + a(x^*)y'(x^*) - b(x^*)\Pi(x^*) \\ &> 0, \end{aligned}$$

which is a contradiction. It follows that $\Pi(x^*) \geq 0$ and thus $\Pi(x) \geq 0$, $\forall x \in [0,1]$.

Furthermore, we have

**Lemma 1.2** *Let* $y(x)$ *be the solution of the problem (1), then we have*

$$\|y\| \leq b^{-1}\|f\| + \max(|\alpha_0|, |\alpha_1|).$$

**Proof.** We construct two barrier functions $\Pi^{\pm}$ defined by

$$\Pi^{\pm}(x) = b^{-1}\|f\| + \max(|\alpha_0|, |\alpha_1|) \pm y(x).$$

Then we have

$$\Pi^{\pm}(0) = b^{-1}\|f\| + \max(|\alpha_0|, |\alpha_1|) \pm y(0) = b^{-1}\|f\| + \max(|\alpha_0|, |\alpha_1|) \pm \alpha_0 \geq 0,$$

$$\Pi^{\pm}(1) = b^{-1}\|f\| + \max(|\alpha_0|, |\alpha_1|) \pm y(1) = b^{-1}\|f\| + \max(|\alpha_0|, |\alpha_1|) \pm \alpha_1 \geq 0$$

and

$$\begin{aligned} \mathcal{L}\Pi^{\pm}(x) &= \varepsilon(\Pi^{\pm}(x))'' + a(x)(\Pi^{\pm}(x))' - b(x)\Pi^{\pm}(x) \\ &= -b(x)(b^{-1}\|f\| + \max(|\alpha_0|, |\alpha_1|)) \pm \mathcal{L}y(x) \\ &= -b(x)[b^{-1}\|f\| + \max(|\alpha_0|, |\alpha_1|)] \pm f(x) \\ &\leq 0. \end{aligned}$$

Therefore, from Lemma 1.1, we obtain $\Pi^{\pm}(x) \geq 0$ for all $x \in [0,1]$, which gives the required estimate.

The above problem is called singularly perturbed due to the fact that the solution $y(x)$ undergoes very rapid changes across very small region(s) (when $\varepsilon \to 0$). These small narrow regions are usually referred to as boundary layers or shock layers in fluid mechanics.

Alternatively, we can define the singular perturbation problem as follows: Consider problem (1) which we denoted by $P_\varepsilon$ and denote its solution by $y_\varepsilon$. When $\varepsilon = 0$, we denote the reduced problem by $P_{reduced}$ and its solution by $y_{reduced}$. Then, the problem $P_\varepsilon$ is called singular perturbation problem if

$$\lim_{\varepsilon\to 0} y_\varepsilon \neq y_{reduced}$$

otherwise, it is called a regular perturbation problem.

Such problem frequently occurs in many real life situations, e.g., in Fluid Mechanics and Solid Mechanics, where we call these layers as boundary layers and edge layers, respectively.

In Figure 1, we plot the left and the right boundary layer functions $\exp(-x/\varepsilon)$ and $\exp(-(1-x)/\varepsilon)$, respectively, for three different values of $\varepsilon$. We took 100 grid points in each case. It can be seen that the layers become very sharp when $\varepsilon \to 0$.

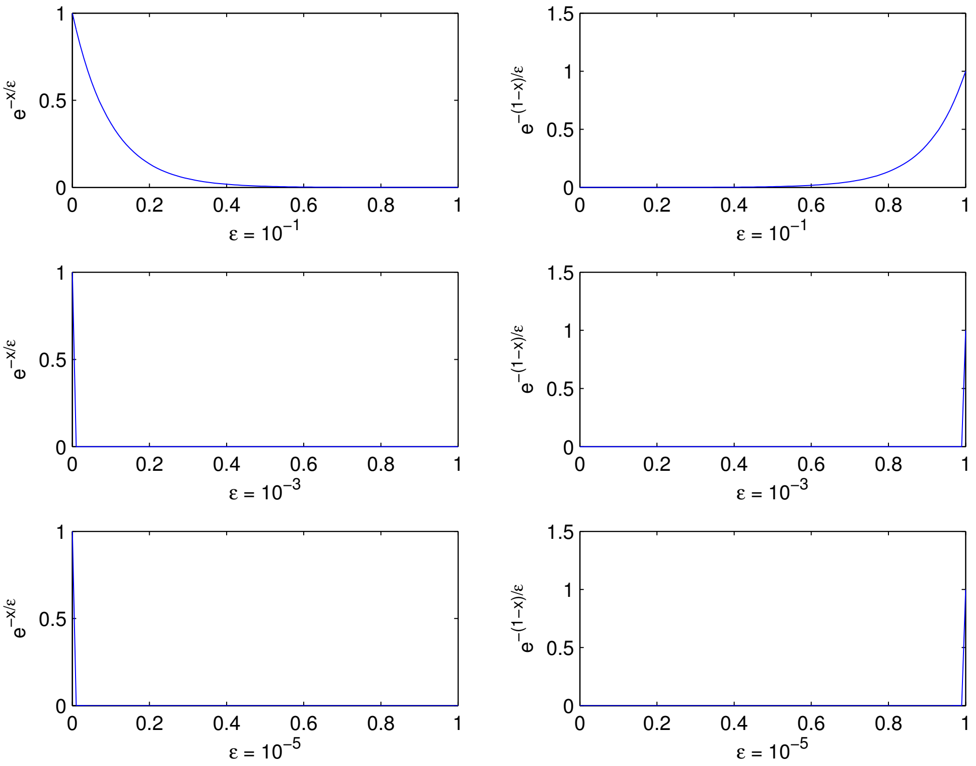


Figure 1: Left and right boundary layer functions

Very often, the parameter dependent differential equations (*e.g.*, (1)) have been solved numerically via various numerical methods (as and when such equations can not be solved analytically), e.g., finite difference methods, finite element methods, spline approximation methods, etc. Unless modified properly, such standard methods often fail in providing reliable numerical results for a range of values of $\varepsilon$. It is possible to design the methods which give numerical results for one fixed $h$ and a given $\varepsilon$, but the same method may not give the similar results if $\varepsilon$ is changed. More precisely, we can say that such method are not $\varepsilon$-uniform in the sense that the parameter $\varepsilon$ and $h$ can not vary independently.

Our aim in this paper is to construct and implement a method which is $\varepsilon$-uniform. This method (which belongs to the class of finite difference methods) is denoted by "Fitted Mesh Finite Difference Method (FMFDM)"as it fits on a specially designed mesh. This special mesh is a piecewise uniform mesh of Shishkin type. The situation when the mesh is uniform throughout the region, the above method will be referred to as "Standard Finite Difference Method (SFDM)." This type of mesh was initially introduced by Shishkin in [15] in order to obtain $\varepsilon$-uniform results. A mesh of this kind uses a fine equidistant mesh inside each layer, and a coarse, again equidistant, mesh on the outside. The Shishkin meshes are discussed in detail in [12, 18] as well as in [13] where, Roos surveyed results on layer-adapted meshes. For a comprehensive list of works on some of this and other techniques, the readers may refer to [4, 5]. A very good discussion on this mesh can be found in Miller et al. [7]. Other notable articles for the problems of the type (1) are those of Boglaev [1], Miller [6], Niijima [9], O'Riordan and Stynes [10], Roos [11], Schatz and Wahlbin [14], Stojanovic [16, 17], etc.

The rest of the paper is organized as follows. In Section 2, we describe how to find the location of the layer whereas in Section 3, we determine the approximate width of the layer. This information about the width of the layer is then used in Section 4 to develop the appropriate mesh. On this mesh, we construct and analyze the finite difference method to solve problem (1). Numerical results with this method and its standard analogue are presented in Section 5. Finally, Section 6 deals with the summary of this work and some future goals.

## 2 Location of the Layer

The location of the boundary layer is one of the important issues related to the singular perturbation problems. It is not always obvious where the boundary and/or interior layer will be located. There are two approaches [8], namely, the analytic and the asymptotic approach to determine the location of the layer. Though our goal (of determining the location of the layer) can be fulfilled just by one of the approaches but we still describe both of these approaches as they do provide additional information about the qualitative behavior of the solution.

### 2.1 Analytic Approach

If we know the analytical solution then we just need to find that which of the following is wrong:

$$\lim_{\epsilon\to 0}\lim_{x\to 0} y(x,\epsilon) = \lim_{x\to 0}\lim_{\epsilon\to 0} y(x,\epsilon)$$

or

$$\lim_{\epsilon\to 0}\lim_{x\to 1} y(x,\epsilon) = \lim_{x\to 1}\lim_{\epsilon\to 0} y(x,\epsilon).$$

If the first of the above conditions holds but not the second one then one may expect the layer at the right end. On the other hand, if the second condition holds whereas the first one does not then the layer may occur near the left end of the interval $[0,1]$. Note that both the conditions will never hold, and if by incidence it happens then the problem will not be singularly perturbed. Moreover, if both the conditions do not hold (for example, the case when the first derivative term is absent in the differential equation (1)) then one may have two boundary layers, one at each end.

### 2.2 Asymptotic Approach

To avoid unnecessary complications, let us consider just the constant coefficient homogeneous problem corresponding to (1)-(2), viz.,

$$\varepsilon y''(x) + ay'(x) - by(x) = 0 \tag{3}$$

and

$$y(0) = \alpha_0 \quad \text{and} \quad y(1) = \alpha_1. \tag{4}$$

Plugging the straightforward expansion

$$y_{as} := y = \sum_{j=0}^{k} \varepsilon^j y_j(x) \ \text{ where } k \in \mathbb{Z}^+ \text{ is finite,} \tag{5}$$

into (3)-(4), we obtain

$$\varepsilon\frac{d^2}{dx^2}\left(\sum_{j=0}^{k}\varepsilon^j y_j(x)\right) + a\frac{d}{dx}\left(\sum_{j=0}^{k}\varepsilon^j y_j(x)\right) - b\left(\sum_{j=0}^{k}\varepsilon^j y_j(x)\right) = 0 \tag{6}$$

and

$$\sum_{j=0}^{k} \varepsilon^j y_j(0) = \alpha_0 \ , \quad \sum_{j=0}^{k} \varepsilon^j y_j(1) = \alpha_1. \tag{7}$$

It is to be noted that $k = 1$ is sufficient in our case.

Sometimes, we may denote this asymptotic solution by $y_{as}$.

Equating like powers of $\varepsilon$ on both sides in (6)-(7), we obtain

terms of order $\varepsilon^0$ :

$$ay_0^{'}(x) - by_0(x) = 0 \tag{8}$$

and

$$y_0(0) = \alpha_0, \ \ y_0(1) = \alpha_1. \tag{9}$$

terms of order $\varepsilon^1$ :

$$y_0''(x) + ay_1'(x) - by_1(x) = 0 \tag{10}$$

and

$$y_1(0) = 0, \ \ y_1(1) = 0. \tag{11}$$

We will consider only two term expansion from (5) as we do not require more terms for the desired explanation in this section.

The general solution of Eq. (8) is given by

$$y_0(x) = A \exp\left(\frac{bx}{a}\right) \tag{12}$$

We note from (9) that there are two boundary conditions on $y_0$, whereas the above general solution contains only one arbitrary constant. Thus $y_0$ may not satisfy both boundary conditions (except by coincidence). To see this, we impose the condition

$$y_0(0) = \alpha_0.$$

Then we obtain from Eq. (12) that

$$A = \alpha_0 \tag{13}$$

so that

$$y_0 = \alpha_0 \exp\left(\frac{bx}{a}\right). \tag{14}$$

On the other hand, imposing the boundary condition

$$y_0(1) = \alpha_1$$

in Eq. (12), we obtain

$$A = \alpha_1 \exp\left(-\frac{b}{a}\right) \tag{15}$$

so that

$$y_0(x) = \alpha_1 \exp\left(-\frac{(1-x)b}{a}\right). \tag{16}$$

Comparing equations (13) and (15), we find that the boundary conditions demand two different values for $A$, namely,

$$A = \alpha_0 \;\text{ and }\; A = \alpha_1 \exp\left(-\frac{b}{a}\right)$$

which are inconsistent unless it happens by coincidence that

$$\alpha_0 = \alpha_1 \exp\left(-\frac{b}{a}\right).$$

Furthermore comparing Eq. (8) with Eq. (3), we observe that the order of the differential equation is reduced from second order (which can cope with two boundary values) to the first order (which can cope with only one boundary condition). Therefore, one of the boundary conditions cannot be satisfied and must be dropped. Dropping the boundary condition $y(0) = \alpha_0$ we conclude that $A = \alpha_1 \exp(-b/a)$ and therefore $y_0$ is given by (16).

Using (16), Eq. (10) becomes

$$y_1'(x) - \frac{b}{a} y_1(x) = -\alpha_1 \left(\frac{b^2}{a^3}\right) \exp\left(\frac{(x-1)b}{a}\right). \tag{17}$$

Multiplying both sides of the above equation by $\exp(-bx/a)$ and simplifying, we obtain

$$\frac{d}{dx}\left(\exp\left(-\frac{b}{a}x\right) y_1(x)\right) = -\alpha_1 \left(\frac{b^2}{a^3}\right) \exp\left(-\frac{b}{a}\right). \tag{18}$$

Integrating and simplifying further, we obtain

$$y_1(x) = -\alpha_1 \left(\frac{b^2}{a^3}\right) x \exp\left(\frac{(x-1)b}{a}\right) + K \exp\left(\frac{bx}{a}\right). \tag{19}$$

Now $y_1$ in (19) contains only one arbitrary constant and therefore cannot cope with the two associated boundary conditions mentioned in (11). As before, again dropping the boundary condition $y_1(0) = 0$ and using the boundary condition $y_1(1) = 0$ we find that $K = \alpha_1 \left(b^2/a^3\right) \exp(-b/a)$, which gives

$$y_1(x) = \alpha_1 \left(\frac{b^2(1-x)}{a^3}\right) \exp\left(\frac{(x-1)b}{a}\right). \tag{20}$$

It then follows that the asymptotic solution of Eq. (3) is

$$y_{as}(x) = \alpha_1 \exp\left(\frac{(x-1)b}{a}\right) + \alpha_1 \varepsilon \left(\frac{b^2}{a^3}\right) (1-x) \exp\left(\frac{(x-1)b}{a}\right) + \ldots \tag{21}$$

Substituting $x = 0$ and $x = 1$ in the above expression, we see that

$$y_{as}(0) = \alpha_1 \left[1 + \varepsilon \left(\frac{b^2}{a^3}\right)\right] \exp\left(-\frac{b}{a}\right) \;\text{ and }\; y_{as}(1) = \alpha_1$$

whereas we have from (4) that

$$y(0) = \alpha_0 \;\text{ and }\; y(1) = \alpha_1.$$

This implies that the two solutions (asymptotic and exact) match at $x = 1$ but not at $x = 0$. This predicts the existence of a boundary layer near the left end of the interval $[0, 1]$.

## 3 Width of the Layer

The width of the layer (denoted by $\delta$) will be of the order of $\varepsilon^\nu$ if the stretching transformation is

$$\xi = \frac{x}{\varepsilon^\nu} \text{ or } \xi = \frac{1-x}{\varepsilon^\nu}$$

where $\nu$ is a real positive constant yet to be determined.

Based on the analysis of the previous section, we know that the boundary layer is located near the left end. Therefore, we consider the stretching transformation

$$\xi = \frac{x}{\varepsilon^\nu},$$

which implies that

$$\frac{d}{dx} \equiv \frac{1}{\varepsilon^\nu}\frac{d}{d\xi} \text{ and } \frac{d^2}{dx^2} \equiv \frac{1}{\varepsilon^{2\nu}}\frac{d^2}{d\xi^2}.$$

It follows then that Eq. (3) is equivalent to

$$\varepsilon^{1-2\nu}\frac{d^2y}{d\xi^2} + a\varepsilon^{-\nu}\frac{dy}{d\xi} - by = 0. \tag{22}$$

We are interested in the matchable expansions of asymptotic solution also known as outer expansion $y^0$ of Eq. (21) which satisfies the right end boundary condition and the inner expansion $y^i$ which satisfies the boundary condition at the left end.

It follows easily from Eq. (21) that the outer expansion is

$$y^o(\xi) = \alpha_1 \exp\left(-\frac{(1-\varepsilon^\nu\xi)b}{a}\right) + \varepsilon\alpha_1\left(\frac{b^2}{a^3}\right)(1-\varepsilon^\nu\xi)\exp\left(-\frac{(1-\varepsilon^\nu\xi)b}{a}\right) + \ldots \tag{23}$$

which when expanded for small value of $\varepsilon$ (keeping $\xi$ fixed), yields the following outer solution in terms of inner variable

$$(y^o)^i = \alpha_1\left[1 + \frac{b}{a}\varepsilon^\nu\xi\right]\exp\left(-\frac{b}{a}\right) + \ldots. \tag{24}$$

On the other hand, we have three possibilities for $\nu$ in (22), viz., $\nu > 1$, $\nu < 1$ and $\nu = 1$. We discuss each one of them below.

1. If $\nu > 1$, then

$$\frac{d^2y^i}{d\xi^2} = 0$$

as the dominant part in Eq. (22). Hence

$$y^i(x) = A + B\frac{x}{\varepsilon^\nu},$$

where A and B are real constants.

Using the associated boundary condition we find that

$$y^i(x) = \alpha_0 + B\frac{x}{\varepsilon^\nu}$$

which when expanded for small value of $\varepsilon$ (keeping $x$ fixed) yields

$$(y^i)^o = \begin{cases} \alpha_0 & \text{if } B = 0 \\ Bx/\varepsilon^\nu & \text{if } B \neq 0 \end{cases} \tag{25}$$

Comparing equations (24) and (25) we find that the two equation are equal only if $\alpha_0 = \alpha_1 \exp(-b/a)$ and $B = 0$, which does not hold in general. This means equations (24) and (25) are not matchable if $\nu > 1$.

2. If $\nu < 1$, then

$$\frac{dy^i}{d\xi} = 0$$

as the dominant part in Eq. (22). Proceeding as above, we find that

$$y^i = \alpha_0$$

is the general solution and hence

$$\left(y^i\right)^o = \alpha_0. \tag{26}$$

Comparing Eq. (26) with Eq. (24) we find that the two equation are equal only if $\alpha_0 = \alpha_1 \exp(-b/a)$ which also does not hold in general. Thus again the two equations are not matchable.

3. If $\nu = 1$, then

$$\frac{d^2y^i}{d\xi^2} + a\frac{dy^i}{d\xi} = 0$$

as the dominant part in Eq. (22). The general solution in this case is given by

$$y^i(x) = (\alpha_0 - B) + B\exp\left(-\frac{ax}{\varepsilon}\right)$$

which when expanded for small value of $\varepsilon$ (keeping $x$ fixed), yields

$$(y^i)^0(x) = \alpha_0 - B. \tag{27}$$

Comparing Eq. (27) with Eq. (24), we find that the two solutions are equal if

$$\alpha_0 - B = \alpha_1 \exp\left(-\frac{b}{a}\right) \quad \text{or} \quad B = \alpha_0 - \alpha_1 \exp\left(-\frac{b}{a}\right)$$

which holds in general and therefore the associated inner expansion can be re-written as

$$y^i(\xi) = \alpha_1 \exp\left(-\frac{b}{a}\right) + \left(\alpha_0 - \alpha_1 \exp\left(-\frac{b}{a}\right)\right)\exp(-a\xi).$$

The above analysis gives the information that $\xi = x/\varepsilon$ is the proper choice for the inner variable because it gave an inner expansion that overlaps the outer expansion. This also suggests us that the width of the layer is of the order of $\varepsilon$.

## 4 Construction and Analysis of the Numerical Method

Let $n$ be a positive integer and recall that $\delta$ denotes the width of the boundary layer which we obtain as $\delta = C\varepsilon$ for some suitable $C > 0$. We denote by $\tau$ the transition parameter.

Since the layer is located near the left end, we divide the interval $[0, 1]$ into two sub-intervals:

$$[0, 1] := [0, \tau] \cup [\tau, 1].$$

The piecewise uniform mesh in these sub-intervals is designed as follows:
Assuming that $n = 2^m$ with $m \geq 3$ an integer, the intervals $(0, \tau)$ and $(\tau, 1)$ are each divided into $n/2$ equal mesh elements.
The parameter $\tau$ is then chosen as

$$\tau = \min\{1/2, \delta\}. \tag{28}$$

Let $x_{j_0} = \tau$ and

$$[0,1] := 0 = x_0 < x_1 < \ldots < x_{j_0} < \ldots < x_n = 1,$$

with $h_j = x_j - x_{j-1}$, where the mesh spacing is given by

$$h_j = \begin{cases} 2\tau n^{-1}, & j = 1, \ldots, j_0, \\ 2(1-\tau)n^{-1}, & j = j_0 + 1, \ldots n. \end{cases} \tag{29}$$

Let us denote the numerical solution vector by $\nu$. Following [7], the usual forward and backward difference operators are then defined as

$$D^+\nu_j = \frac{\nu_{j+1} - \nu_j}{h_{j+1}}$$

$$D^-\nu_j = \frac{\nu_j - \nu_{j-1}}{h_j}$$

and the central difference operator is given by

$$\delta_0\nu_j = \frac{2\,(D^+\nu_j - D^-\nu_j)}{h_j + h_{j+1}}.$$

Denoting the piecewise approximations of the functions $a(x)$, $b(x)$ and $f(x)$ in the intervals $[x_{j-1}, xj]$ for each $j = 1(1)n-1$ by $a_j$, $b_j$ and $f_j$, respectively, the desired fitted mesh finite difference scheme for problem (1) is given by

$$\left.\begin{array}{c} \varepsilon\delta_0\nu_j + a_j D^+\nu_j - b_j\nu_j = f_j,\ j = 1(1)n-1 \\ \nu_0 = \alpha_0,\ \nu_n = \alpha_1. \end{array}\right\} \tag{30}$$

**Remark 4.1** When $h_{j+1} = h_j = h = 1/n$, then scheme (30) yields

$$\left.\begin{array}{c} \varepsilon\frac{\nu_{j+1} - 2\nu_j + \nu j - 1}{h^2} + a_j\frac{\nu_{j+1} - \nu_j}{h} - b_j\nu_j = f_j,\ j = 1(1)n-1 \\ \nu_0 = \alpha_0,\ \nu_n = \alpha_1. \end{array}\right\} \tag{31}$$

We refer to this scheme as "Standard Finite Difference Method(SFDM)" whereas we call (30) as the "Fitted Mesh Finite Difference Methods (FMFDM)".

Re-writing the scheme (30) in form of the system of equations

$$A\nu = F, \tag{32}$$

we see that the local truncation error $\tau_j(y)$ is given by

$$\tau_j(y) = (Ay)_j - F_j = (A(y - \nu))_j.$$

Thus

$$\max_j |y_j - \nu_j| \quad \leq \quad \|A^{-1}\| \max_j |\tau_j(y)|. \tag{33}$$

The inequality (33) trivially holds for the case when $j = 0$ or $n$. On the other hand, when $1 \leq j \leq n-1$, we have

$$\tau_j(y) \quad = \quad T_0 y_j + T_1 y_j^{'} + T_2 y^{''}(\xi_j), \tag{34}$$

where $\xi_j \in (x_{j-1}, x_{j+1})$ and

$$\begin{aligned} T_0 &= \left(r_j^- + r_j^c + r_j^+\right) + q_j b_j, \\ T_1 &= h_{j+1} r_j^+ - h_j r_j^- - q_j a_j, \\ T_2 &= \tfrac{h_{j+1}^2}{2!} r_j^+ + \tfrac{h_j^2}{2!} r_j^- - \varepsilon q_j. \end{aligned} \tag{35}$$

Since the mesh is piecewise uniform with mesh spacing $\widetilde{h}$ (where $\widetilde{h}$ is $h_{j+1}$ or $h_j$), one obtains via (34)-(35)

$$\max_{1 \leq j \leq n-1} |\tau_j(y)| \leq M \widetilde{h} y^{''}(\xi_j), \tag{36}$$

where $\xi_j \in (x_{j-1}, x_{j+1})$.

Moreover, by a result in [19], we have

$$\|A^{-1}\| \leq \max_{1 \leq j \leq n-1} \left\{ |r_j^c| - \left(|r_j^-| + |r_j^+|\right) \right\}^{-1} \leq M. \tag{37}$$

Using (36) and (37) in (33), we obtain

$$\max_{1 \leq j \leq n-1} |y_j - \nu_j| \leq M \widetilde{h} y^{''}(\xi_j), \tag{38}$$

We consider now two cases: either the grid point $x_j$ is inside the fine mesh, i.e.,

$$j \in \{n - j_0 + 1, \ldots, n-1\} \tag{39}$$

or it is inside the coarse mesh, i.e.,

$$j \in \{1, \ldots, n - j_0\}. \tag{40}$$

This leads to

$$\widetilde{h} = \begin{cases} 4\left(\frac{n}{4}\right)^{-1} \varepsilon \ln n, & j = 1, \ldots, j_0, \\ \frac{2}{n} - 4\left(\frac{n}{4}\right)^{-1} \varepsilon \ln n, & j = j_0 + 1, \ldots n. \end{cases} \tag{41}$$

**Lemma 4.2** *The solution $y(x)$ of (1) admits the decomposition*

$$y(x) := v_\varepsilon(x) + w_\varepsilon(x),$$

*where the regular(smooth) component $v_\varepsilon(x) (\equiv y_r(x))$ satisfies*

$$|v_\varepsilon(x)| \leq M \left[1 + \exp\left(-\frac{ax}{\varepsilon}\right)\right],$$

$$\left|v_\varepsilon^{(k)}(x)\right| \le M\left[1+(\varepsilon)^{2-k}\exp\left(-\frac{ax}{\varepsilon}\right)\right],\ \forall\, k \ge 1$$

*and the singular component* $w_\varepsilon(x)(\equiv y_s(x))$ *satisfies*

$$\left|w_\varepsilon^{(k)}(x)\right| \le M\,(\varepsilon)^{-k}\exp\left(-\frac{ax}{\varepsilon}\right),\ \forall\, k \ge 0.$$

Using (41), distinguishing between regular and singular components, and simplifying further, we obtain the following main result

**Theorem 4.3** *Let* $a(x) \ge a > 0$, $b(x) \ge b > 0$ *and* $f(x)$ *be sufficiently smooth functions so that* $y(x) \in C^2[0,1]$. *Let* $\nu_j,\ j = 0(1)n$, *be the approximate solution of (1), obtained using FMFDM with* $\nu_0 = y(0)$, $\nu_n = y(1)$. *Then, there exists a constant* $M$ *independent of* $\varepsilon$ *and the mesh size such that*

$$\sup_{0<\varepsilon\le 1}\ \max_{0<j\le n} |y_j - \nu_j| \le M n^{-1}\ln n.$$

## 5 Numerical Results

We test the scheme (30) for problem (1) with following cases of the coefficient functions:

**Example 5.1** Consider

$$a(x) = 1 + x^2,\ b(x) = \exp(x)$$

and

$$f(x) = \frac{1}{1-\exp(-2/\varepsilon)}\left[\frac{\exp(-2x/\varepsilon)}{\varepsilon}\big(4 - 2(1+x^2) - \varepsilon\exp(x)\big) + \exp(x - 2/\varepsilon)\right].$$

The exact solution corresponding to this problem is given by

$$y(x) = \frac{\exp(-2x/\varepsilon) - \exp(-2/\varepsilon)}{1-\exp(-2/\varepsilon)}.$$

**Example 5.2** Consider

$$a(x) = (x+1)^3,\ b(x) = 0$$

Here $f(x)$ is such that the exact solution is given by

$$y(x) = \frac{\exp\left\{-[(x+1)^4 - 1]/4\varepsilon\right\}}{(x+1)^3} + \exp\left(-x/2\right).$$

Having $\nu_j \equiv \nu_j^n$ (the numerical solution) for different values of $n$ and $\varepsilon$, the maximum errors (denoted by $E_{n,\varepsilon}$) at all the mesh points are evaluated using the formula

$$E_{n,\varepsilon} := \max_{0\le j\le n} |y(x_j) - \nu_j|$$

and are presented in Tables 1-4. Further, we compute

$$E_n = \max_{0<\varepsilon\le 1} E_{n,\varepsilon}.$$

Table 1: Maximum Errors: Uniform Mesh (SFDM): Example 5.1

| $\varepsilon$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ | $n = 2048$ | $n = 4096$ |
|---|---|---|---|---|---|---|
| $5^{-1}$ | 0.80E-02 | 0.40E-02 | 0.20E-02 | 0.10E-02 | 0.50E-03 | 0.25E-03 |
| $5^{-2}$ | 0.44E-01 | 0.23E-01 | 0.12E-01 | 0.58E-02 | 0.29E-02 | 0.15E-02 |
| $5^{-3}$ | 0.20E+00 | 0.11E+00 | 0.57E-01 | 0.29E-01 | 0.15E-01 | 0.75E-02 |
| $5^{-4}$ | 0.17E+00 | 0.26E+00 | 0.23E+00 | 0.13E+00 | 0.70E-01 | 0.36E-01 |
| $5^{-5}$ | 0.39E-01 | 0.75E-01 | 0.14E+00 | 0.23E+00 | 0.26E+00 | 0.17E+00 |
| $5^{-6}$ | 0.81E-02 | 0.16E-01 | 0.32E-01 | 0.61E-01 | 0.12E+00 | 0.20E+00 |
| $5^{-7}$ | 0.16E-02 | 0.33E-02 | 0.65E-02 | 0.13E-01 | 0.26E-01 | 0.50E-01 |
| $5^{-8}$ | 0.32E-03 | 0.65E-03 | 0.13E-02 | 0.26E-02 | 0.52E-02 | 0.10E-01 |
| $5^{-9}$ | 0.65E-04 | 0.13E-03 | 0.26E-03 | 0.52E-03 | 0.10E-02 | 0.21E-02 |
| $10^{-7}$ | 0.13E-04 | 0.25E-04 | 0.51E-04 | 0.10E-03 | 0.20E-03 | 0.41E-03 |
| $10^{-8}$ | 0.13E-05 | 0.25E-05 | 0.51E-05 | 0.10E-04 | 0.20E-04 | 0.41E-04 |
| $10^{-9}$ | 0.13E-06 | 0.25E-06 | 0.51E-06 | 0.10E-05 | 0.20E-05 | 0.41E-05 |
| $10^{-10}$ | 0.13E-07 | 0.25E-07 | 0.51E-07 | 0.10E-06 | 0.20E-06 | 0.41E-06 |

Table 2: Maximum Errors: Piecewise Uniform Mesh (FMFDM): Example 5.1

| $\varepsilon$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ | $n = 2048$ | $n = 4096$ |
|---|---|---|---|---|---|---|
| $5^{-1}$ | 0.80E-02 | 0.40E-02 | 0.20E-02 | 0.10E-02 | 0.50E-03 | 0.25E-03 |
| $5^{-2}$ | 0.44E-01 | 0.23E-01 | 0.12E-01 | 0.58E-02 | 0.29E-02 | 0.15E-02 |
| $5^{-3}$ | 0.79E-01 | 0.51E-01 | 0.31E-01 | 0.18E-01 | 0.11E-01 | 0.60E-02 |
| $5^{-4}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $5^{-5}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $5^{-6}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $5^{-7}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $5^{-8}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $5^{-9}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $10^{-7}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $10^{-8}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $10^{-9}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $10^{-10}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $10^{-11}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $10^{-12}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $10^{-13}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $10^{-14}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $10^{-15}$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |
| $E_n$ | 0.80E-01 | 0.51E-01 | 0.31E-01 | 0.19E-01 | 0.11E-01 | 0.60E-02 |

## 6 Summary and Future Directions

In this paper, we have designed and implemented a numerical method for solving one dimensional singular perturbation problems. The method is tested for two examples and the $\varepsilon$-uniform numerical results are obtained as shown in Tables 2 and 4.

The results are compared with the standard finite difference method (in our case the FMFDM with $h_{j+1} = h_j = h = 1/n$) and the outcomes are self-explanatory. As mentioned in the introduction section, one may observe that for the SFDM, the errors are smaller for many combinations of $n$ and $\varepsilon$, however, these errors do not remain same for a fixed $n$ and various $\varepsilon$ values. However, we

Table 3: Maximum Errors: Uniform Mesh (SFDM): Example 5.2

| $\varepsilon$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ | $n = 2048$ | $n = 4096$ |
|---|---|---|---|---|---|---|
| $5^{-1}$ | 0.99E-02 | 0.50E-02 | 0.25E-02 | 0.13E-02 | 0.63E-03 | 0.32E-03 |
| $5^{-2}$ | 0.36E-01 | 0.19E-01 | 0.95E-02 | 0.48E-02 | 0.24E-02 | 0.12E-02 |
| $5^{-3}$ | 0.13E+00 | 0.76E-01 | 0.41E-01 | 0.22E-01 | 0.11E-01 | 0.56E-02 |
| $5^{-4}$ | 0.16E+00 | 0.20E+00 | 0.16E+00 | 0.91E-01 | 0.50E-01 | 0.26E-01 |
| $5^{-5}$ | 0.39E-01 | 0.75E-01 | 0.14E+00 | 0.20E+00 | 0.18E+00 | 0.10E+00 |
| $5^{-6}$ | 0.87E-02 | 0.16E-01 | 0.32E-01 | 0.61E-01 | 0.12E+00 | 0.19E+00 |
| $5^{-7}$ | 0.24E-02 | 0.36E-02 | 0.67E-02 | 0.13E-01 | 0.26E-01 | 0.50E-01 |
| $5^{-8}$ | 0.11E-02 | 0.10E-02 | 0.15E-02 | 0.27E-02 | 0.53E-02 | 0.10E-01 |
| $5^{-9}$ | 0.83E-03 | 0.51E-03 | 0.45E-03 | 0.62E-03 | 0.11E-02 | 0.21E-02 |
| $10^{-7}$ | 0.77E-03 | 0.41E-03 | 0.24E-03 | 0.20E-03 | 0.25E-03 | 0.43E-03 |
| $10^{-8}$ | 0.76E-03 | 0.38E-03 | 0.20E-03 | 0.11E-03 | 0.68E-04 | 0.65E-04 |
| $10^{-9}$ | 0.76E-03 | 0.38E-03 | 0.19E-03 | 0.97E-04 | 0.50E-04 | 0.28E-04 |
| $10^{-10}$ | 0.76E-03 | 0.38E-03 | 0.19E-03 | 0.96E-04 | 0.48E-04 | 0.24E-04 |

Table 4: Maximum Errors: Piecewise Uniform Mesh (FMFDM): Example 5.2

| $\varepsilon$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ | $n = 2048$ | $n = 4096$ |
|---|---|---|---|---|---|---|
| $5^{-1}$ | 0.99E-02 | 0.50E-02 | 0.25E-02 | 0.13E-02 | 0.63E-03 | 0.32E-03 |
| $5^{-2}$ | 0.36E-01 | 0.19E-01 | 0.95E-02 | 0.48E-02 | 0.24E-02 | 0.12E-02 |
| $5^{-3}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.80E-02 | 0.45E-02 |
| $5^{-4}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $5^{-5}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $5^{-6}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $5^{-7}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $5^{-8}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $5^{-9}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $10^{-7}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $10^{-8}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $10^{-9}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $10^{-10}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $10^{-11}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $10^{-12}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $10^{-13}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $10^{-14}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $10^{-15}$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |
| $E_n$ | 0.57E-01 | 0.37E-01 | 0.23E-01 | 0.14E-01 | 0.79E-02 | 0.45E-02 |

can see the $\varepsilon$-uniform behavior in the results corresponding to FMFDM.

It is worthwhile to mention here that the continuous problem satisfies a continuous minimum principle whereas the discrete problem does not. It is interesting to note (as is also pointed out in [2]-[3] and [7]) that the lack of a discrete minimum/maximum principle does not destroy the $\varepsilon$-uniform convergence and this is due to the fact that we have used appropriate fitted mesh.

As the title reads, the asymptotic analysis helped us in determining the location of the layer with which we were able to design a condensed mesh in the layer region. The idea of asymptotic analysis is applied for a problem in which the layer is located near only one end. However there are other possibilities, for example, either $a(x) \equiv 0$ or $a(x)$ changes sign. While in the former case, we will have two boundary layers (one at each end), the later case may lead to turning point problems. Though linear, still these turning point problems are challenging ones as the location of the layers demand rigorous analysis (as in Section 3) in order to identfy appropriate distinguished limits which provide the width of the internal layer(s). These issues (both theoretically and numerically) are currently being investigated by the authors.

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