European Society of Computational Methods in Sciences and Engineering (ESCMSE)

Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM)
vol. 18, no. 1-4, 2025, pp. 19-28
ISSN 1790–8140

Hyers–Ulam Stability of ψ-Hilfer Nonlinear Differential Equations of Complex Order

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Received 18 June, 2025; accepted in revised form 30 June, 2025

Abstract: The primary purpose of this work is to study complex-order differential equations. In particular, we explore results involving Hilfer fractional derivatives of complex order. We develop existence and uniqueness results for the proposed problem using the Schauder Fixed Point Theorem. Furthermore, we investigate the stability of the Hyers-Ulam solution under suitable conditions. In addition, we present an example to illustrate the applicability of the results obtained. These findings lay a foundation for future research on advanced fractional models with applications in science and engineering, particularly in systems exhibiting memory, hereditary properties, or complex dynamic behavior. The developed framework may also guide the formulation of numerical methods and stability analysis in practical problems ranging from viscoelastic materials to anomalous diffusion and control theory.

Keywords: Hyers–Ulam Stability, Caputo derivative, ψ-Hilfer Fractional Derivative, Complex Order Fractional Derivatives, Nonlinear Fractional Differential Equations, Fixed Point Methods, Existence and Uniqueness Results.

Mathematics Subject Classification: 26E30, 34M10, 34A08, 30D35, 47H10

1 Introduction

Calculus plays a central role in the development of analysis, and differentiation is one of its most important components. While classical differential operators rely on positive integer orders, fractional calculus extends the concept by allowing the order of derivation to be a non-integer, thereby introducing memory and hereditary effects into the model. Fractional calculus (FC), known as non-integer-order calculus, traces its origins back to 1695. The concept emerged from the correspondence between Leibniz and Bernoulli on the meaning of a derivative of order $\frac{1}{2}$ applied to a power function. In a brilliant and arguably prophetic note, Leibniz provided the correct result and remarked, "...the paradox would one day have several important consequences." More than three centuries later, FC has become a rich field of scientific inquiry, offering significant theoretical and

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applied advances and sparking ongoing debate in light of recent developments [11]. Several definitions of fractional derivatives, such as the Riemann–Liouville, Caputo, and Grunwald–Letnikov operators, have been introduced to capture different geometric and analytical properties.

In efforts to model real-world processes using fractional-order differential equations [14], researchers have frequently encountered the challenge that the fractional derivative order α does not remain constant. Instead, it may vary within an interval, from 0 to 1, 1 to 2, or even across the entire range from 0 to 2. Various approaches have been proposed to address this phenomenon. The next extension of fractional-order derivatives is variable-order fractional derivatives (VOFDs), in which the order of differentiation changes over time and/or depends on spatial coordinates. This concept extends fractional-order methods and provides greater accuracy than non-integer-order methods. When simulating complex phenomena in biological and engineering systems where memory effects or diffusion behavior vary over time or space, variable-order fractional derivatives (VOFDs) are essential [8]. VOFDs offer a more flexible and accurate representation of evolving processes than classical and constant-order fractional models [4, 13], allowing the differentiation order to change dynamically.

A natural extension of this idea is to consider complex-order derivatives, where the order may be taken as $m + ni$ or $\alpha = \alpha_1 + i\alpha_2$ with $i^2 = -1$. The imaginary component of the order introduces oscillatory memory effects, enabling the modeling of systems in which hereditary behavior and oscillations coexist. This provides additional flexibility compared to real-order derivatives, leading to integral kernels with richer spectral characteristics and allowing the description of physical phenomena such as viscoelastic materials, anomalous diffusion with wave-like behavior, and electromagnetic processes with memory. Moreover, combining fractional operators with respect to functions and weighted fractional operators yields new classes of generalized ψ-fractional operators. In particular, the ψ-Hilfer derivative of complex order provides an extended analytical framework capable of capturing non-local behavior, functional dependence, and oscillatory effects, making it a powerful tool in modern fractional calculus.

In recent years, complex-order derivatives have been widely used to model physical phenomena exhibiting non-monotonic relaxation behavior due to their potential. Complex-order derivatives extend the modeling capabilities of non-integer-order derivatives by capturing oscillatory and memory-dependent dynamics. E.R. Love first discussed the study of imaginary order derivatives [7], laying the foundation for further exploration in this field. Complex-order differential equations have been effectively applied to model anomalous diffusion processes exhibiting spatial and temporal complexities [1]. They play a key role in characterizing the viscoelastic behavior of materials, where higher-order differentiation provides a more realistic representation of the combined elastic and viscous responses [12]. This method improves our understanding of the mechanical characteristics of materials such as biological tissues and polymers [5].

In [15], the author studied a coupled fractional boundary value problem system subject to integral boundary conditions. They used fixed-point theorems to investigate the existence and uniqueness of solutions to the coupled system. In this work, the authors discussed positive solutions and provided an example of theoretical applicability. Su [10] discussed the boundary value problem for a fractional-order coupled differential system. The author used the Riemann–Liouville fractional derivative to deal with the fractional differential operator. This study established the existence and uniqueness of the solution using the Schauder fixed-point theorem after converting the problem into a Fredholm integral equation.

The main task of dealing with the ψ-HCO-NL-DEs in this study is to generalize and unify fractional differential operators, which is crucial because it provides greater flexibility when modeling systems with intricate memory and heredity characteristics. These models apply to signal processing, anomalous diffusion, and viscoelasticity, where systems display nonlocal and oscillatory behavior. More accurate explanations of real-world phenomena that are not well captured by classical or ordinary fractional models are made possible by the combination of complex order and the ψ-Hilfer derivative. Examining the existence, uniqueness, and stability of solutions provides a strong theoretical basis for effective numerical techniques and simulations in the applied sciences.

We are motivated to study differential equations of complex order, as discussed above. This study becomes more important for exploring new theories and analyses related to the proposed model. We are interested in exploring the results associated with the existence of a solution to the proposed problem and, if the solution exists, under what conditions it becomes unique? For this purpose, we provide a uniqueness inequality under suitable conditions by which we can ensure the uniqueness of complex-order differential equations (CODEs) with the Hilder derivative. Furthermore, we extend our study to examine the solution's stability using Hyers-Ulam stability. A crucial aspect of the approximate solution is that it remains close to the exact one for the considered problem, and that it is closer to the exact solution. We can ensure this by using an inequality supported by this stability theory. Overall, This study focuses on the existence, uniqueness, and Hyers–Ulam stability of the proposed ψ-Hilfer complex-order operator.

Consider the following ψ-Hilfer complex-order nonlinear differential equations (ψ-HCO-DEs) of the form:

$$\begin{aligned} D^{\alpha_1,\alpha_2;\psi}\mathcal{U}(t) &= \mathcal{F}\left(t,\mathcal{U}(t), D^{\beta}\mathcal{U}(t)\right), \quad t \in \mathcal{I} = (a,b], \\ \mathcal{J}^{1-\alpha;\psi}\mathcal{U}(t)\big|_{t=a} &= \mathcal{U}_a, \ \alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2, \end{aligned} \tag{1}$$

where $D^{\alpha_1,\alpha_2;\psi}$ is the ψ-Hilfer complex-order derivative of order $\alpha_1 = \alpha_1' + i\alpha_2'$ and $\alpha_2 = \alpha_1'' + i\alpha_2''$, and D^{β} denotes the Caputo fractional derivative of order β. Here, $0 < \mathcal{R}(\alpha_1) < 1$, $0 \leq \mathcal{R}(\alpha_2) \leq 1$, and $0 < \beta \leq 1$, where $\alpha_1', \alpha_2', \alpha_1''$, and α_2'' are real constants. Let $\mathbb{R}$ be a Banach space, and let $\mathcal{F} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function.

The following table presents the description of all the symbols used throughout the paper.

Table 1: List of main symbols

Symbol	**Description**
$I = (a,b]$	Time interval of the problem (1).
$\psi(t)$	Increasing function defining the ψ-Hilfer complex-order operators.
$\alpha = \alpha_1 + i\alpha_2$	Complex fractional order.
$\alpha_1 = \alpha_1' + i\alpha_2'$	First complex-order parameter in the ψ-Hilfer derivative.
$\alpha_2 = \alpha_1'' + i\alpha_2''$	Second complex-order parameter in the ψ-Hilfer derivative.
$0 < R(\alpha_1) < 1$	Real part of α_1.
$0 \leq R(\alpha_2) \leq 1$	Real part of α_2.
β	Order of the Caputo derivative, $0 < \beta \leq 1$.
$D^{\alpha;\psi}$	ψ–Riemann–Liouville fractional derivative of complex order).
$D^{\alpha_1,\alpha_2;\psi}$	ψ–Hilfer fractional derivative of complex order.
$\mathcal{J}^{1-\alpha;\psi}$	ψ–Riemann–Liouville fractional integral of complex order.
$\mathcal{U}(t)$	Unknown solution.
$\mathcal{U}_a$	Initial condition.
$\mathcal{F}\left(t,\mathcal{U}(t), D^{\beta}\mathcal{U}(t)\right)$	Nonlinear term defining the differential equation.
$C(\mathcal{I}$	Space of continuous functions on I.
$C_{1-\mu,\psi}(\mathcal{I})$	Weighted space: $(\psi(t) - \psi(a))^{\mu-1}F(t) \in C(I)$.
$\mu = R(\alpha)$	Real part of the fractional order.
$\Gamma(\cdot)$	Gamma function.
$\rho(t)$	Weight function for generalized HU and HUR stability.
B_f, $B_{f,\rho}$	Constants in HU and HUR stability.

2 Preliminaries

This section presents some important definitions and lemmas that will be used to investigate the main results of this work.

Let us consider the following function space: suppose $C(\mathcal{I})$ denotes the space of continuous functions from the interval $\mathcal{I}$ into $\mathbb{R}$, equipped with the norm

$$\|x\|_C = \max\Big\{|x(t)| : t \in \mathcal{I}\Big\}.$$

The weighted space $C_{1-\mu,\psi}(\mathcal{I})$, of function $\mathcal{F}$ on $\mathcal{I}$ is defined by

$$C_{1-\mu,\psi}(\mathcal{I}) = \Big\{\mathcal{F} : \mathcal{I} \to \mathbb{R} : (\psi(t) - \psi(a))^{\mu-1}\mathcal{F}(t) \in C(\mathcal{I})\Big\},\ 0 \leq \mu(= \mathcal{R}(\alpha) < 1),$$

with the norm

$$\|\mathcal{F}\|_{C_{1-\mu,\psi}} = \Big\|(\psi(t) - \psi(a))^{\mu-1}\mathcal{F}(t)\Big\|_{C[a,b]} = \max_{t\in\mathcal{I}}\Big|(\psi(t) - \psi(a))^{\mu-1}\mathcal{F}(t)\Big|.$$

Definition 1. The $\psi-$ Riemann-Liouville (R-L) fractional integral of order $\alpha \in\in \mathbb{C},\ (\mathcal{R}(\alpha) > 0)$ of a function $\mathcal{F}$ is defined by,

$$\mathcal{J}^{\alpha;\psi}\mathcal{F}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\psi(t) - \psi(a))^{\alpha-1}\mathcal{F}(s)ds,\ t \geq 0. \tag{2}$$

Definition 2. The $\psi-$R-L fractional derivative of order $\alpha \in\in \mathbb{C},\ (\mathcal{R}(\alpha) > 0)$ of a function $\mathcal{F}$ is defined by,

$$\mathcal{D}^{\alpha;\psi}\mathcal{F}(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\int_0^t \psi'(\psi(t) - \psi(a))^{n-\alpha-1}\mathcal{F}(s)ds,\ t \geq 0, \tag{3}$$

where $n = [\mathcal{R}(\alpha)] + 1$.

Definition 3. The $\psi-$Caputo fractional derivative of order $\alpha \in\in \mathbb{C},\ (\mathcal{R}(\alpha) > 0)$ of a function $\mathcal{F}$ is defined by,

$$\mathcal{D}^{\alpha;\psi}\mathcal{F}(t) = \mathcal{J}^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\mathcal{F}(t),\ t \geq 0, \tag{4}$$

where $n = [\mathcal{R}(\alpha)] + 1$.

Definition 4. The $\psi-$Hilfer fractional derivative (HFD) of $0 < \alpha_1 < 1$ and $0 \leq \alpha_2 \leq 1$ of function $\mathcal{F}(t)$ is defined by

$$\mathcal{D}^{\alpha_1,\alpha_2;\psi}\mathcal{F}(t) = \mathcal{J}^{\alpha_2(1-\alpha_1);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathcal{J}^{(1-\alpha_2)(1-\alpha_1);\psi}\mathcal{F}(t),\ t \geq 0 \tag{5}$$

The $\psi-$HFD as above defined, can be written in the following

$$\mathcal{D}^{\alpha_1,\alpha_2;\psi}\mathcal{F}(t) = \mathcal{J}^{\alpha-\alpha_1;\psi}\mathcal{D}^{\alpha;\psi}\mathcal{F}(t).$$

Remark 1. (a) If $\alpha_2 = 0$ $(\alpha_1'' = 0, \alpha_2'' = 0)$, then $\mathcal{D}^{\alpha_1,\alpha_2;\psi} = \mathcal{D}^{\alpha_1,0;\psi}$ is called the RL fractional derivative of complex order.
(b) If $\alpha_2 = 1$ $(\alpha_1'' = 1, \alpha_2'' = 0)$, then $\mathcal{D}^{\alpha_1,\alpha_2;\psi} = \mathcal{J}^{1-\alpha_1;\psi}\mathcal{D}^{1;\psi}$ is called the Caputo fractional

derivative of complex order.

Definition 5. The stirling asymptotic formula of gamma function for $\mathcal{Z} \in \mathbb{C}$ is following

$$\Gamma(\mathcal{Z}) = \sqrt{2\pi}\, \mathcal{Z}^{\mathcal{Z}-\frac{1}{2}} e^{-\mathcal{Z}} \left[1 + O\left(\frac{1}{\mathcal{Z}}\right)\right], \quad |\arg(\mathcal{Z})| < \pi,\ |\mathcal{Z}| \to \infty, \tag{6}$$

and its result for $|\Gamma(\mathcal{Z}_1 + i\mathcal{Z}_2)|,\ \mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{R}$ is

$$|\Gamma(\mathcal{Z}_1 + i\mathcal{Z}_2)| = \sqrt{2\pi}\, |\mathcal{Z}_2|^{\mathcal{Z}_1-\frac{1}{2}} e^{-\mathcal{Z}_1-\frac{\pi|\mathcal{Z}_2|}{2}} \left[1 + O\left(\frac{1}{|\mathcal{Z}_2|}\right)\right], \quad |\mathcal{Z}_2| \to \infty. \tag{7}$$

We now introduce the concepts of Hyers–Ulam (HU) stability and Hyers–Ulam–Rassias (HUR) stability for ψ-Hilfer complex-order differential equations (ψ-HCO-DEs).

Definition 6. Let $\epsilon > 0$ be a positive real number and $\rho : \mathcal{I} \to \mathbb{R}^+$ be a continuous function. The analysis begins with the following set of inequalities:

$$\left| D^{\alpha_1,\alpha_2;\psi}\mathcal{U}(t) - \mathcal{F}\left(t, \mathcal{U}(t), D^{\beta}\mathcal{U}(t)\right) \right| \le \epsilon, t \in \mathcal{I}, \tag{8}$$

$$\left| D^{\alpha_1,\alpha_2;\psi}\mathcal{U}(t) - \mathcal{F}\left(t, \mathcal{U}(t), D^{\beta}\mathcal{U}(t)\right) \right| \le \epsilon\rho(t), t \in \mathcal{I}, \tag{9}$$

$$\left| D^{\alpha_1,\alpha_2;\psi}\mathcal{U}(t) - \mathcal{F}\left(t, \mathcal{U}(t), D^{\beta}\mathcal{U}(t)\right) \right| \le \rho(t), t \in \mathcal{I}. \tag{10}$$

Definition 7. The ψ-HCO-DEs (1) is HU stable if there exists a real number $B_f > 0$ such that for each $\epsilon > 0$ and for each solution $\mathcal{V} \in C_{1-\mu,\psi}(\mathcal{I})$ of the inequlaity (8) there exists a constant $\mathcal{U}$ of problem (1) with

$$\left| \mathcal{V}(t) - \mathcal{U}(t) \right| \le B_f \epsilon, t \in \mathcal{I}. \tag{11}$$

Definition 8. The ψ-HCO-DEs (1) is generalized HU stable if there exist $\rho \in C_{1-\mu,\psi}(\mathcal{I}),\ \rho_f(0) = 0$ such that for each solution $\mathcal{V} \in C_{1-\mu,\psi}(\mathcal{I})$ of the inequality (8) there exists a solution $\mathcal{U} \in C_{1-\mu,\psi}(\mathcal{I})$ of problem (1) with

$$\left| \mathcal{V}(t) - \mathcal{U}(t) \right| \le \rho_f \epsilon, t \in \mathcal{I}. \tag{12}$$

Definition 9. The ψ-HCO-DEs (1) is HUR stable with respect to $\rho \in C_{1-\mu,\psi}(\mathcal{I})$ if there exists a real number $B_{f,\,\rho} > 0$ such that for each $\epsilon > 0$ and for each solution $\mathcal{V} \in C_{1-\mu,\psi}(\mathcal{I})$ of the inequality (9) there exists a solution $\mathcal{U} \in C_{1-\mu,\psi}(\mathcal{I})$ of the problem (1) with

$$\left| \mathcal{V}(t) - \mathcal{U}(t) \right| \le B_{f,\,\rho}\, \rho(t) \epsilon, t \in \mathcal{I}. \tag{13}$$

Definition 10. The ψ-HCO-DEs (1) is generalized HUR stable with respect to $\rho \in C_{1-\mu,\psi}(\mathcal{I})$ if there exists a real number $B_{f,\rho} > 0$ such that for each solution $\mathcal{V} \in C_{1-\mu,\psi}(\mathcal{I})$ of the inequality (10) there exists a solution $\mathcal{U} \in C_{1-\mu,\psi}(\mathcal{I})$ of the problem (1) with

$$\left| \mathcal{V}(t) - \mathcal{U}(t) \right| \le B_{f,\,\rho}\, \rho(t), t \in \mathcal{I}. \tag{14}$$

Lemma 1. A fucntion $\mathcal{U}$ is the solution of

$$\begin{aligned} D^{\alpha_1,\alpha_2;\psi}\mathcal{U}(t) &= \mathcal{F}(t), \quad t \in \mathcal{I} = (a, b], \\ \mathcal{J}^{1-\alpha;\psi}\mathcal{U}(t)\big|_{t=a} &= \mathcal{U}_a,\ \alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2, \end{aligned} \tag{15}$$

equivalent to the solution of an integral equation

$$\mathcal{U}(t)=\frac{\mathcal{U}_a}{\Gamma(\alpha)}(\psi(t)-\psi(a))^{\alpha-1}+\frac{1}{\Gamma(\alpha_1)}\int_a^t\psi'(s)(\psi(t)-\psi(a))^{\alpha_1-1}\mathcal{F}(s)ds. \tag{16}$$

Remark 1. A function $\mathcal{V}\in C_{1-\mu,\psi}(\mathcal{I})$ is a solution of the inequality

$$\left|D^{\alpha_1,\alpha_2;\psi}\mathcal{V}(t)-\mathcal{F}\left(t,\mathcal{V}(t),D^{\beta}\mathcal{V}(t)\right)\right|\le\epsilon, t\in\mathcal{I} \tag{17}$$

iff there exists a function $F\in C_{1-\mu,\psi}(\mathcal{I})$ such that

(i) $|F(t)|\le\epsilon, t\in\mathcal{I}$.

(ii) $D^{\alpha_1,\alpha_2;\psi}\mathcal{U}(t)=\mathcal{F}\left(t,\mathcal{U}(t),D^{\beta}\mathcal{U}(t)\right)+F(t), t\in\mathcal{I}$.

(iii) If $\mathcal{V}$ is a solution of the inequality (8), then $\mathcal{V}$ is a solution of the following integral inequality

$$\left|\mathcal{V}(t)-\frac{\mathcal{V}_a}{\Gamma(\alpha)}(\psi(t)-\psi(a))^{\alpha-1}-\frac{1}{\Gamma(\alpha_1)}\int_a^t\psi'(s)(\psi(t)-\psi(a))^{\alpha_1-1}\mathcal{F}\left(s,\mathcal{V}(s),D^{\beta}\mathcal{V}(s)\right)ds\right|\le\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha|\Gamma(\alpha_1)|}\epsilon.$$

Lemma 2. Suppose $\vartheta(=\mathcal{R}(\alpha)>0)$, $a(t)$ is a non-negative locally integrable function on $a\le t\le b$(some $b\le\infty$) and let $F(t)$ be a non-negative locally integrable on $a\le t\le b$ function with

$$|\mathcal{U}(t)|\le a(t)+F(t)\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\vartheta-1}\mathcal{U}(s)ds, t\in\mathcal{I} \tag{18}$$

with some $\vartheta>0$. Then

$$|\mathcal{U}(t)|\le a(t)+\int_a^t\left[\sum_{n=1}^{\infty}\frac{(F(s)\Gamma(\vartheta))^n}{\Gamma(n\vartheta)}\psi'(s)(\psi(t)-\psi(s))^{n\vartheta-1}\right]a(s)ds, t\in[a,b]. \tag{19}$$

Theorem 1. (Banach Fixed Point Theorem [12, 4]) Let $\mathcal{C}$ be a Banach space, and $\mathcal{D}\subset\mathcal{C}$ be a closed non-empty subset. Suppose that $\mathcal{P}:\mathcal{D}\to\mathcal{D}$ is a strict contraction; that is, there exists a constant $k\in(0,1)$ such that

$$\|\mathcal{P}(x)-\mathcal{P}(y)\|\le k\|x-y\|,\quad\text{for all } x,y\in\mathcal{D},$$

then $\mathcal{P}$ has a unique fixed point in $\mathcal{D}$.

Theorem 2. (Schauder Fixed Point Theorem [12, 4]) Let $\mathcal{C}$ be a Banach space and and $\mathcal{D}\subset\mathcal{C}$ be a closed and bounded nonempty convex subset and $\mathcal{T}:\mathcal{D}\to\mathcal{D}$ be a compact and continuous map. Then the operator $\mathcal{T}$ has at least one fixed point in $\mathcal{D}$. We first present some preliminary findings to facilitate the development of our main results. The inequalities below will be essential in subsequent proofs.

Thoerem 3. Let $\mathcal{U}\in C^1([0,a])$, and let the Caputo fractional derivative of order $0<\beta\le1$ be defined by

$$D^{\beta}\mathcal{U}(t)=\frac{1}{\Gamma(1-\beta)}\int_0^t(t-s)^{-\beta}\mathcal{U}'(s)\,ds.$$

Then there exist constants $\mathcal{M}_1,\mathcal{M}_2>0$ such that

$$|D^{\beta}\mathcal{U}(t)|\le\mathcal{M}_1\|\mathcal{U}\|_{C^1},\ t\in[0,a]$$

and

$$|D^{\beta}\mathcal{U}_1(t) - D^{\beta}\mathcal{U}_2(t)| \le \mathcal{M}_2\|\mathcal{U}_1 - \mathcal{U}_2\|_{C^1}, t \in [0, a],$$

where $\|\mathcal{U}\|_{C^1} := \max\{\|\mathcal{U}\|_\infty, \|\mathcal{U}'\|_\infty\}$.
Proof. Let $\mathcal{U}' \in C([0, a])$, we have

$$\begin{aligned} |D^{\beta}\mathcal{U}(t)| & \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}|\mathcal{U}'(s)|\, ds \\ & \le \frac{\max_{s\in[0,t]} |\mathcal{U}'(s)|}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} ds \\ & = \frac{\|\mathcal{U}'\|}{\Gamma(1-\beta)} \frac{t^{1-\beta}}{1-\beta} \\ & = \frac{t^{1-\beta}}{\Gamma(2-\beta)}\|\mathcal{U}'\| \\ & \le \frac{a^{1-\beta}}{\Gamma(2-\beta)}\|\mathcal{U}'\| \\ & \le \mathcal{M}_1\|\mathcal{U}\|_{C^1},, \end{aligned} \tag{20}$$

where $\mathcal{M}_1 := \frac{a^{1-\beta}}{\Gamma(2-\beta)}$.
Finally, we have

$$|D^{\beta}\mathcal{U}(t)| \le \mathcal{M}_1\|\mathcal{U}\|_{C^1}.$$

Now,

$$\begin{aligned} |D^{\beta}\mathcal{U}_1(t) - D^{\beta}\mathcal{U}_2(t)| & \le \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \max_{t\in[0,t]} |\mathcal{U}_1'(s) - \mathcal{U}_2'(s)|\, ds \\ & \le \frac{\|\mathcal{U}_1' - \mathcal{U}_2'\|a^{1-\beta}}{\Gamma(2-\beta)} \\ & \le \mathcal{M}_2\|\mathcal{U}_1 - \mathcal{U}_2\|_{C^1}, \end{aligned} \tag{21}$$

where $\mathcal{M}_2 := \frac{a^{1-\beta}}{\Gamma(2-\beta)}$.
Hence

$$|D^{\beta}\mathcal{U}_1(t) - D^{\beta}\mathcal{U}_2(t)| \le \mathcal{M}_2\|\mathcal{U}_1 - \mathcal{U}_2\|_{C^1}.$$

3 Main Results

In this section, we present the following results based on fixed point theorems. The main investigations are outlined as follows:

1. We establish the inequality for the uniqueness of the ψ-HCO-DEs (1) using the Banach fixed point theorem.

2. We use the Schauder fixed-point theorem to develop the existence of solutions.

3. Finally, we investigate stability theories utilizing Hyers-Ulam-type stabilities for the given problem ψ-HCO-DEs (1).

4. The results obtained are validated with an illustrative example.

Methodological Note. In this work, we employ tools from fixed point theory, specifically, the Banach and Schauder fixed point theorems—to establish the existence and uniqueness of solutions. We also use fractional calculus techniques, including ψ-Hilfer and Caputo derivatives of complex order. To investigate stability, we adopt the Hyers–Ulam stability definition framework. These methods provide a rigorous mathematical foundation for analyzing the behavior of solutions under perturbations.

4 Illustrative Example

Consider the following ψ-Hilfer complex-order nonlinear differential equation defined on the interval $\mathcal{I} = (0, 1]$:

$$D^{\alpha_1,\alpha_2;\psi}\mathcal{U}(t) = \mathcal{F}(t, \mathcal{U}(t), D^{\beta}\mathcal{U}(t)), \tag{22}$$

with initial condition:

$$\mathcal{J}^{1-\alpha;\psi}\mathcal{U}(t)\Big|_{t=0} = \mathcal{U}_0, \quad \alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2, \tag{23}$$

where:

$$\alpha_1 = 0.5 + 0.1i, \quad \alpha_2 = 0.3, \quad \beta = 0.5, \quad \psi(t) = t, \quad \mathcal{U}_0 = 1,$$

and the nonlinear term is given by:

$$\mathcal{F}(t, \mathcal{U}(t), D^{\beta}\mathcal{U}(t)) = \mathcal{U}(t).$$

This example satisfies the conditions stated in our main results. Specifically:

- The function $\mathcal{F}$ is continuous and satisfies a Lipschitz condition in $\mathcal{U}$.
- The initial condition and fractional orders are chosen to meet the assumptions required for applying Schauder and Banach fixed point theorems.
- Therefore, the existence and uniqueness of the solution, as well as its Hyers–Ulam stability, are guaranteed under the theoretical framework developed in this paper.

5 Conclusion

In this study, we introduced a new class of problems involving an extension of noninteger-order differential equations called complex-order differential equations. We derived several fundamental results using the Hilfer fractional derivative within the functional space $C_{1-\mu;\psi}$.

Under appropriate assumptions, we established conditions that ensure the existence and uniqueness of solutions to the ψ-Hilfer complex-order nonlinear differential equations (ψ-HCO-DEs). In addition, we examined the Hyers–Ulam stability of the solutions, demonstrating the robustness of the proposed framework.

All theoretical findings were supported by a detailed illustrative example, which confirmed the applicability and correctness of the results.

This research lays the groundwork for further studies on complex-order nonlinear systems involving generalized fractional operators. Future investigations may extend these results to systems with variable fractional order or those defined over multidimensional domains, making the models more representative of real-world phenomena.

In particular, the developed framework has potential for applications in modeling.

- Viscoelastic materials,

- Diffusion processes in heterogeneous media and
- Control systems with memory effects.

Furthermore, one can develop numerical methods suitable for engineering and biological simulations based on the theoretical stability conclusions presented, thereby increasing the study's practical value.

6 Future Work

While the present work is devoted entirely to analytical aspects, specifically the existence, uniqueness, and Hyers–Ulam stability of ψ-Hilfer complex-order differential equations, several primary directions can be pursued next. Namely, to provide numerical and semi-analytical methods that can find approximate solutions to the suggested complex-order operators. It is applicable when oscillatory memory effects or non-local interactions are significant. Another interesting direction is to apply the ψ-Hilfer complex-order operators to real physical models, such as viscoelastic behavior [2], odd diffusion processes, and control theory systems with memory, where the interaction between hereditary effects and oscillations is essential. Extending the current analysis to variable-order complex derivatives or higher-dimensional problems could also broaden the modeling potential of these operators [3]. Recent research has demonstrated that complex-order derivatives can be applied in various fields of biology and engineering, including robotic movement, controller design, fuel cell system modeling, and the study of infectious disease propagation [6]. Complex-order structures have been used in genetic algorithm-based modeling, transfer function modeling, numerical optimization, and controller tuning. Subsequent research has developed complex-order control schemes for nonlinear systems, analyzed their stability and frequency response characteristics, and demonstrated the benefits of complex-order formulations in real-world applications. Based on these developments, the ψ-Hilfer complex-order formulation introduced in this paper could be adapted in future research to areas such as advanced controller design, identification of nonlinear systems, and biological modeling domains where oscillatory memory behavior is important. In addition, utilizing the analytical results of this study in conjunction with modern numerical optimization methods or complex-order controller structures (such as PI, PID, FOPID, or COPID controllers [9]) is an interesting area for further research.

Acknowledgment

This research was funded in whole by the National Science Center, Poland, grant number 2023/51/B/ST8/01062.

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