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An Iterative Inverse-free Method to Solve the Matrix Equation $X + A^*X^{-1}A = I$

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Abstract: In this paper, we present a novel inverse-free iterative method to compute the maximal positive definite solution for the matrix equation $X + A^*X^{-1}A = I$. The proposed method generates a rapidly converging sequence directed towards the inverse matrix, using a fixed point method formed by a specific type of polynomials, thereby facilitating the estimation of the maximal positive definite solution while adhering to a defined stopping criterion. We provide a thorough proof of convergence for the proposed methodology, outlining the necessary conditions and the convergence rate required to effectively solve the matrix equation. We show that the sequence converging quadratically to the inverse matrix plays a significant role in enhancing the convergence rate of the method, thereby improving its overall effectiveness in solving the matrix equation. In the numerical results section, we consider several matrices of varying sizes and scenarios, paying particular attention to computational cost and execution time. Furthermore, the numerical evaluations demonstrate that our method outperforms alternative approaches, especially in reducing the total number of matrix-matrix multiplications and minimizing execution time.

Keywords: Matrix equation, Iterative method, Inverse-free, Maximal positive definite solution, Matrix-matrix multiplications.

Mathematics Subject Classification: 65F10 , 65F30, 65H10, 15A24, 93B40

1 Introduction

The nonlinear matrix equation $X + A^*X^{-1}A = Q$, in which Q is a positive definite Hermitian matrix, has many applications in the fields of control theory, ladder networks analysis, dynamic programming, stochastic filtering, and statistics [1, 12]. Without loss of generality, the above equation can be written in the form

$$X + A^*X^{-1}A = I, \tag{1.1}$$

in which I is the identity matrix. An efficient approach to this end is to use Cholesky decomposition $Q = LL^*$ and insert it into $\tilde{X} = L^{-1}XL^{-*}$ and $\tilde{A} = L^{-1}AL^{-*}$ [8]. This nonlinear matrix

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equation (1.1) can be considered as a natural generalization of the scalar equation $x+a^2/x=1$, or $\phi(x)=a^2$, where $\phi(x)=x(1-x)$. Note that the latter equation has a solution $x\in(0,1)$ whenever $a^2\leqslant\max\phi(x)$ holds.
If we put $F=0$, $B=I$, and $R=0$ in the time-discrete Riccati algebraic equation

$$Q+F^*XF-X-(F^*XB+A^*)(R+B^*XB)^{-1}(B^*XF+A)=0,$$

then it becomes evident that the equation (1.1) is a special case of the Riccati equation. It is shown (see [18] and references therein) that the equation (1.1) has a positive definite solution $X=W^*W$ if and only if

$$A=W^*Z, \tag{1.2}$$

where W is a nonsingular matrix and the columns of $\binom{W}{Z}$ are orthonormal. Any other positive definite solution can also be expressed in this way. In addition, it is proved that if the equation (1.1) has a positive definite solution, then it has one largest positive definite solution X_+ and one smallest positive definite solution X_-, meaning that every positive definite solution X holds in $X_-\leqslant X\leqslant X_+$. In the case where A is invertible, it is shown in [4] that X is the solution of $X+A^*X^{-1}A=I$ if and only if $Y=I-X$ is the solution of the equation $Y+A^*Y^{-1}A=I$. In particular, if Y_+ is a maximal positive definite solution, then $X_-=I-Y_+$ will be a minimal positive definite solution.

A natural approach to find the maximal solution of equation (1.1) is

$$\begin{cases} X_0=I, \\ X_{n+1}=I-A^*X_n^{-1}A, \quad n=0,1,\ldots. \end{cases} \tag{1.3}$$

It is shown in [4] that $X_0\leqslant X_1\leqslant\ldots$ and $\lim_{n\to\infty}X_n=X_+$ hold. In addition, it is proved [17] that $\|X_{n+1}-X_+\|\leqslant\|X_+^{-1}A\|^2\|X_n-X_+\|$, and $\rho(X_+^{-1}A)\leqslant1$, where $\rho(\cdot)$, the spectral radius, is the maximum of the absolute values of eigenvalues of a square matrix.
Also, due to numerical problems, inverse-free methods have recently received much attention, all of which have the approximation of X_n^{-1} in common. For example, Zhang [17] used an iteration of Newton's method to approximate X_n^{-1} and proposed the following method:

$$\begin{cases} X_0=Y_0=I, \\ X_{n+1}=I-A^*Y_nA, \\ Y_{n+1}=Y_n(2I-X_nY_n), \quad n=0,1,\ldots. \end{cases} \tag{1.4}$$

Although he showed $X_0\leqslant X_1\leqslant\ldots$, $Y_0\geqslant Y_1\geqslant\ldots$, and $\lim_{n\to\infty}X_n=X_+$, he did not express the rate of convergence of the method. Transforming the calculations, Guo and Lancaster [10] modified Zhang's method as follows:

$$\begin{cases} X_0=Y_0=I, \\ Y_{n+1}=Y_n(2I-X_nY_n), \\ X_{n+1}=I-A^*Y_{n+1}A, \quad n=0,1,\ldots. \end{cases} \tag{1.5}$$

They showed that the algorithm (1.5) performs twice as fast as the algorithm (1.4) does, and if $\rho(X_+^{-1}A)<1$ holds, then the algorithm (1.5) is R-linearly convergent. To obtain a more accurate algorithm than Zhang's algorithm and a less expensive algorithm than that of Guo and Lancaster, El-Sayed and Al-Dbiban [3] proposed the following algorithm to calculate the maximal solution of equation (1.1):

$$\begin{cases} X_0=Y_0=I, \\ Y_{n+1}=(I-X_n)Y_n+I, \\ X_{n+1}=I-A^*Y_{n+1}A, \quad n=0,1,\ldots. \end{cases} \tag{1.6}$$

In [7], Esmaeili and Pirnia presented a second-order convergent method to approximate the inverse matrix. Using their method, the following algorithm can be used to calculate the maximal solution of equation (1.1):

$$\begin{cases} X_0 = Y_0 = I, \\ Y_{n+1} = Y_n \left(5.5I - X_n Y_n (8I - 3.5 X_n Y_n)\right), \\ X_{n+1} = I - A^* Y_{n+1} A, \quad n = 0, 1, \dots. \end{cases} \tag{1.7}$$

Also, Erfanifar, Sayevand, and Esmaeili [5] proposed the following method to calculate the maximal solution of equation (1.1) and proved that it is linearly convergent:

$$\begin{cases} X_0 = Y_0 = I, \\ Y_{n+1} = -I + Y_n \left(3I + X_n - 2X_n Y_n\right), \\ X_{n+1} = I - A^* Y_{n+1} A, \quad n = 0, 1, \dots. \end{cases} \tag{1.8}$$

Li and Li [13] proposed a third-order convergent method for inverse matrix approximation, which leads to the following method to calculate the maximal solution of equation (1.1):

$$\begin{cases} X_0 = Y_0 = I, \\ Y_{n+1} = Y_n \left(3I + (-3I + X_n Y_n) X_n Y_n\right), \\ X_{n+1} = I - A^* Y_{n+1} A, \quad n = 0, 1, \dots. \end{cases} \tag{1.9}$$

In [6], and [11–16], authors introduced some methods to solve nonlinear matrix equations.
In the next section, we present a new iterative inverse-free method to calculate the maximal positive definite solution of equation (1.1). It is proved that this method is linearly convergent. Numerical tests show that our method is more efficient than the methods mentioned above.
The following notations are used in this paper. The notation $A \geqslant 0$ $(A > 0)$ means A is a positive semidefinite (positive definite), A^* denotes the conjugate transpose of A, and I is the identity matrix. In addition, $A \geqslant B$ $(A > B)$ is another notation for $A - B \geqslant 0$ $(A - B > 0)$. The norm used in this paper is the spectral norm of matrix A, $\|A\| = \sqrt{\rho(A^*A)}$, unless otherwise is stated.

2 An iterative inverse-free method

In this section, we present a new method that converges to the solution of (1.1) more rapidly than other methods. Like the previously mentioned approaches, our method is iterative and consists of two steps. The initial step of it, derived from a specific type of polynomials, approximates the inverse matrix using the function

$$f(y) = f_1 + f_2 x + f_3 y + f_4 xy + f_5 x^2 y + f_6 xy^2 + f_7 x^2 y^2,$$

that satisfies $f(\frac{1}{x}) = \frac{1}{x}$, yielding $f_3 + f_6 = 1$, $f_1 + f_4 + f_7 = 0$, and $f_2 + f_5 = 0$. Also, using $f'(\frac{1}{x}) = 0$, the first derivative of $f(y)$ in terms of y at $y = \frac{1}{x}$, we have $f_3 + 2f_6 = 0$, $f_4 + 2f_7 = 0$, and $f_5 = 0$. Additionally, at the initial point when $x = 1$ and $y = 1$, we have $f(1) = 1$. Therefore, we can obtain

$$f(y) = 1 + 2y - 2xy - xy^2 + x^2 y^2.$$

In the second step, the next iteration of the sequence is generated. Upon introducing an iterative methodology aimed at solving equation (1.1), we will proceed to investigate the underlying properties associated with this equation. After this analytical exploration, we will establish the convergence criteria of the proposed methodology and delineate the requisite conditions necessary for identifying a solution to the equation.

Now, there is a chance to consider an inverse-free iterative method to obtain the maximal positive definite solution of the equation (1.1) as follows:

$$\begin{cases} X_0 = Y_0 = I, \\ Y_{n+1} = \left(\left((X_n - I)Y_n - 2I\right) X_n + 2I\right) Y_n + I, \\ X_{n+1} = I - A^* Y_{n+1} A, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{2.1}$$

The above algorithm has five matrix-matrix multiplications; therefore, to reduce its computational cost we can minimize the number of matrix-matrix multiplications per iteration. To achieve this goal, it is rewritten (2.1) as follows:

$$\begin{cases} X_0 = Y_0 = I, \\ S_n = X_n Y_n, \\ Y_{n+1} = (S_n - Y_n)(S_n - 2I) + I, \\ X_{n+1} = I - A^* Y_{n+1} A, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{2.2}$$

Each iteration of the proposed algorithm consists of four matrix-matrix multiplications and does not require the inverse matrix calculation.

First, we bring some theorems for necessary and sufficient conditions without proofs.

Theorem 1. *[15] If equation* (1.1) *has a positive definite solution X_+, then $X_+ \leqslant I$.*

Theorem 2. *[15] If equation* (1.1) *has a positive definite solution X_+, then*

(1) $A^*A < I$.

(2) $AA^* < X_+$.

Theorem 3. *[4] If* (1.1) *has a positive definite solution X_+, then $\rho(A) < \frac{1}{2}$.*

Theorem 4. *[2] If $\|A\| < \frac{1}{2}$, then* (1.1) *has a positive definite solution X_+ in $[\frac{1}{2}I, I]$, which is the maximal positive definite solution of* (1.1).

Theorem 5. *[19] Assume that P and C are positive semidefinite matrices. PC is positive semidefinite if and only if $PC = CP$.*

Lemma 1. *[19] If $P \leqslant C$, then $D^*PD \leqslant D^*CD$ for any full rank matrix D.*

Lemma 2. *[17] If C and P are Hermitian matrices of the same order with $P > 0$, then $CPC + P^{-1} \geqslant 2C$.*

Lemma 3. *Assume that D is a positive semidefinite matrix and P and C, with $P \leqslant C$, are Hermitian matrices. If $PD = DP$ and $CD = DC$, then $PD \leqslant CD$.*

Proof. It is clear that $0 \leqslant C - P$. According to Theorem 5, since $(C - P)$ and D are positive semidefinite matrices and $(C-P)D = CD - PD = DC - DP = D(C-P)$, it results in $(C-P)D$ is a positive semidefinite matrix. We have

$$0 \leqslant (C - P)D = CD - PD \quad \Rightarrow \quad PD \leqslant CD.$$

□

Lemma 4. *If A is normal and the two sequences $\{X_n\}$ and $\{Y_n\}$ are generated by the method* (2.2)*, then we have*

$$AY_n = Y_n A, \quad AX_n = X_n A, \quad A^*Y_n = Y_n A^*, \quad A^*X_n = X_n A^*, \quad X_n Y_n = Y_n X_n.$$

Proof. Since $Y_0 = I$ and $X_0 = I$, we have $AY_0 = Y_0A$, $AX_0 = X_0A$, $X_0Y_0 = Y_0X_0$ and also for A^*. Notice that $Y_1 = I$, then $AY_1 = Y_1A$ and it is true for A^*. Since $X_1 = I - A^*A$ and A is normal, resulting in

$$AX_1 = A(I - A^*A) = A - AA^*A = A - A^*AA = (I - A^*A)A = X_1A,$$

and also for A^*. It is clear that $X_1Y_1 = Y_1X_1$. Moreover, for $n = 2$, we have

$$\begin{aligned} Y_2 &= (((X_1 - I)Y_1 - 2I)X_1 + 2I)Y_1 + I \\ &= I + ((-A^*A - 2I)(I - A^*A) + 2I) \\ &= I + A^*A + (A^*A)^2. \end{aligned}$$

Then,

$$\begin{aligned} AY_2 = A(I + A^*A + (A^*A)^2) &= A + AA^*A + A(A^*A)^2 \\ &= (I + A^*A + (A^*A)^2)A = Y_2A, \end{aligned}$$

and also for A^*. Since $X_2 = I - A^*Y_2A$, we have

$$\begin{aligned} AX_2 = A(I - A^*Y_2A) &= A - AA^*Y_2A \\ &= A - A^*Y_2AA = (I - A^*Y_2A)A = X_2A, \end{aligned}$$

and also for A^*. We can obtain,

$$\begin{aligned} X_2Y_2 = (I - A^*Y_2A)Y_2 &= Y_2 - A^*Y_2AY_2 \\ &= Y_2 - Y_2A^*Y_2A = Y_2(I - A^*Y_2A) = Y_2X_2. \end{aligned}$$

Therefore, we show that relations $AY_n = Y_nA$, $AX_n = X_nA$, $X_nY_n = Y_nX_n$, and also for A^* are true for $n = 0, 1, 2$. Considering the induction assumptions for n, we have

$$AY_n = Y_nA, \quad A^*Y_n = Y_nA^*, \quad AX_n = X_nA, \quad A^*X_n = X_nA^*, \quad X_nY_n = Y_nX_n.$$

Now, we show that the above relations are true for $n + 1$. Note that,

$$\begin{aligned} AY_{n+1} &= A((X_nY_n)^2 - Y_nX_nY_n - 2X_nY_n + 2Y_n + I) \\ &= A(X_nY_n)^2 - AY_nX_nY_n - 2AX_nY_n + 2AY_n + A \\ &= (X_nY_n)^2A - Y_nX_nY_nA - 2X_nY_nA + 2Y_nA + A = Y_{n+1}A, \end{aligned}$$

and also it is true for A^*. Then, we have

$$AX_{n+1} = A(I - A^*Y_{n+1}A) = A - AA^*Y_{n+1}A = A - A^*Y_{n+1}AA = X_{n+1}A,$$

and also it holds for A^*. To complete the proof, we obtain

$$\begin{aligned} X_{n+1}Y_{n+1} &= (I - A^*Y_{n+1}A)Y_{n+1} \\ &= Y_{n+1} - A^*Y_{n+1}AY_{n+1} = Y_{n+1} - Y_{n+1}A^*Y_{n+1}A \\ &= Y_{n+1}X_{n+1}. \end{aligned}$$

□

In the following theorem, it is shown that the sequence $\{X_n\}$ is monotone decreasing and convergent to the maximal positive definite solution X_+.

Theorem 6. *If the equation* (1.1) *has a positive definite solution, the two sequences* $\{X_n\}$ *and* $\{Y_n\}$ *are generated by the method* (2.2)*, and* A *is normal, then* $\{X_n\}$ *is monotone decreasing and convergent to the maximal solution* X_+ *and also* $\{Y_n\}$ *is monotone increasing and convergent to* X_+^{-1}*. If the matrix* A *is nonsingular and* $X_n > 0$ *holds for every* n*, then* (1.1) *has a positive definite solution.*

Proof. First, we prove that

$$I = X_0 \geqslant X_1 \geqslant X_2 \geqslant \cdots \geqslant X_n \geqslant X_+, \tag{2.3}$$

and

$$I = Y_0 \leqslant Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_n \leqslant X_+^{-1}. \tag{2.4}$$

Suppose that X_+ is a solution to the equation (1.1), then it results

$$X_+ = I - A^* X_+^{-1} A.$$

So, it can be concluded that $X_0 = I \geqslant X_+$ holds. On the other hand, according to the method (2.2), in the case $n = 1$ it is clear that

$$X_1 = I - A^*A \geqslant I - A^*X_+^{-1}A = X_+.$$

Thus, the inequality $X_0 \geqslant X_1 \geqslant X_+$ is obtained. Since $Y_1 = I$, it is followed that

$$\begin{aligned} X_2 = I - A^*Y_2A &= I - A^*\left((((I - A^*A - I) - 2I)\,(I - A^*A) + 2I) + I\right)A \\ &= I - A^*\left(I + A^*A + (A^*A)^2\right)A \\ &= I - A^*A - A^*(A^*A)A - A^*(A^*A)^2A \\ &= X_1 - A^*(A^*A)A - A^*(A^*A)^2A. \end{aligned}$$

Based on the equation $X_2 = X_1 - A^*(A^*A)A - A^*(A^*A)^2A$ obtained from the above equation, $X_1 \geqslant X_2$ and therefore $X_0 \geqslant X_1 \geqslant X_2$ are acquired. For the sequence $\{Y_n\}$, we have $Y_0 = Y_1 = I$, and since $I \leqslant X_+^{-1}$ holds, it is clear that $Y_0 = Y_1 \leqslant X_+^{-1}$ is resulted. In this case

$$\begin{aligned} Y_0 = Y_1 = I \leqslant& I + A^*A + (A^*A)^2 \\ &= (((X_1 - I)Y_1 - 2I)\,X_1 + 2I)\,Y_1 + I = Y_2, \end{aligned}$$

which requires $Y_0 \leqslant Y_1 \leqslant Y_2$. Since $Y_0 = Y_1 = I$, $X_0 = I$, and $X_1 \leqslant I$, we have $Y_0 \leqslant X_0^{-1}$ and $Y_1 \leqslant X_1^{-1}$. For Y_2, we have

$$Y_2 = (Y_1 - X_1Y_1)(2I - X_1Y_1) + I = (I - X_1)(2Y_1 - Y_1X_1Y_1) + I.$$

According to Lemma 2, we obtain $2Y_1 - Y_1X_1Y_1 \leqslant X_1^{-1}$. Note that, $0 \leqslant 2I - X_1 = 2Y_1 - Y_1X_1Y_1$ and $0 \leqslant I - X_1 = I - X_1Y_1$. Since

$$(I - X_1)(2I - X_1) = 2I - 3X_1 + X_1^2 = (2I - X_1)(I - X_1),$$

and

$$(I - X_1)X_1^{-1} = X_1^{-1} - X_1X_1^{-1} = X_1^{-1} - X_1^{-1}X_1 = X_1^{-1}(I - X_1),$$

based on Lemma 3, we can conclude that $(I - X_1)(2I - X_1) \leqslant (I - X_1)X_1^{-1}$. Thus, from

$$Y_2 = (I - X_1)(2Y_1 - Y_1X_1Y_1) + I \leqslant (I - X_1)X_1^{-1} + I = X_1^{-1} \leqslant X_+^{-1},$$

relations $Y_2 \leqslant X_+^{-1}$ and $Y_2 \leqslant X_1^{-1}$ are obtained. In the case of X_2, the following is obtained

$$X_2 = I - A^* Y_2 A \geqslant I - A^* X_+^{-1} A = X_+,$$

i.e. $X_0 \geqslant X_1 \geqslant X_2 \geqslant X_+$. Note that, since $X_2 \leqslant X_1$, we obtain $Y_2 \leqslant X_1^{-1} \leqslant X_2^{-1}$. So far, it has been shown that the inequalities (2.3), (2.4), and $Y_n \leqslant X_n^{-1}$ are true for $n = 0, 1, 2$. Considering the induction assumption for n, we assume

$$I = X_0 \geqslant X_1 \geqslant X_2 \geqslant \cdots \geqslant X_n \geqslant X_+,$$

and

$$I = Y_0 \leqslant Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_n \leqslant X_+^{-1},$$

are established. Now, it is easy to show that these inequalities are also true for $n+1$. To this end, notice that

$$\begin{aligned} Y_{n+1} &= (((X_n - I)Y_n - 2I)X_n + 2I)Y_n + I \\ &= (X_n Y_n - Y_n - 2I)X_n Y_n + 2Y_n + I, \end{aligned}$$

then

$$\begin{aligned} Y_{n+1} - Y_n &= (X_n Y_n - Y_n - 2I)X_n Y_n + Y_n + I \\ &= (X_n Y_n - I)X_n Y_n - (Y_n + I)X_n Y_n + (Y_n + I) \\ &= X_n Y_n (X_n Y_n - I) + (Y_n + I)(I - X_n Y_n) \\ &= (Y_n + I - X_n Y_n)(I - X_n Y_n). \end{aligned}$$

Since $Y_n \leqslant X_n^{-1}$, the inequality $X_n Y_n \leqslant I$ is derived. Thus, $0 \leqslant I - X_n Y_n$ and $0 \leqslant Y_n + I - X_n Y_n$ hold. According to Lemma 4, we have $X_n Y_n = Y_n X_n$, then

$$\begin{aligned} Y_{n+1} - Y_n &= (Y_n + I - X_n Y_n)(I - X_n Y_n) \\ &= (X_n Y_n)^2 - Y_n X_n Y_n - 2X_n Y_n + Y_n + I \\ &= (X_n Y_n)^2 - X_n Y_n Y_n - 2X_n Y_n + Y_n + I \\ &= (X_n Y_n - I)X_n Y_n - X_n Y_n (Y_n + I) + (Y_n + I) \\ &= (X_n Y_n - I)X_n Y_n + (I - X_n Y_n)(Y_n + I) \\ &= (I - X_n Y_n)(Y_n + I - X_n Y_n). \end{aligned}$$

So, based on Theorem 5, we obtain $0 \leqslant Y_{n+1} - Y_n$ and then $Y_n \leqslant Y_{n+1}$. Note that,

$$Y_{n+1} = (Y_n - X_n Y_n)(2I - X_n Y_n) + I = (I - X_n)(2Y_n - Y_n X_n Y_n) + I.$$

According to Lemma 2, $2Y_n - Y_n X_n Y_n \leqslant X_n^{-1}$ holds. Using Lemma 4, we have

$$(I - X_n)X_n^{-1} = X_n^{-1} - X_n X_n^{-1} = X_n^{-1} - X_n^{-1} X_n = X_n^{-1}(I - X_n),$$

and

$$\begin{aligned} (I - X_n)(2Y_n - Y_n X_n Y_n) &= 2Y_n - Y_n X_n Y_n - 2X_n Y_n + (X_n Y_n)^2 \\ &= 2Y_n - Y_n X_n Y_n - 2Y_n X_n + (Y_n X_n)^2 \\ &= (2Y_n - Y_n X_n Y_n)(I - X_n). \end{aligned}$$

Based on Lemma 3, we obtain $(I - X_n)(2Y_n - Y_nX_nY_n) \leqslant (I - X_n)X_n^{-1}$. For Y_{n+1}, we have

$$\begin{aligned} Y_{n+1} &= (I - X_n)(2Y_n - Y_nX_nY_n) + I \\ &\leqslant (I - X_n)X_n^{-1} + I = X_n^{-1} - X_nX_n^{-1} + I \\ &= X_n^{-1} \leqslant X_+^{-1}, \end{aligned}$$

which means $Y_{n+1} \leqslant X_n^{-1} \leqslant X_+^{-1}$. Considering the sequence $\{X_n\}$, it is obvious that

$$X_n - X_{n+1} = A^*(Y_{n+1} - Y_n)A.$$

Since $Y_n \leqslant Y_{n+1}$, it follows from the above equation that $X_n \geqslant X_{n+1}$. On the other hand,

$$X_{n+1} = I - A^*Y_{n+1}A \geqslant I - A^*X_+^{-1}A = X_+,$$

the result is $X_n \geqslant X_{n+1} \geqslant X_+$. Then, $X_n^{-1} \leqslant X_{n+1}^{-1}$ implies that $Y_{n+1} \leqslant X_{n+1}^{-1}$. This induction is completed for $n + 1$. Therefore,

$$I = X_0 \geqslant X_1 \geqslant X_2 \geqslant \cdots \geqslant X_n \geqslant X_+,$$

and

$$I = Y_0 \leqslant Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_n \leqslant X_+^{-1},$$

and $0 \leqslant I - X_nY_n$ are true in any case $n \geqslant 0$, and there are $\lim_{n\to\infty} Y_n = Y$ and $\lim_{n\to\infty} X_n = X$. Taking limit in the method (2.1) results in

$$\begin{aligned} Y =& \left(\left((X - I)Y - 2I\right)X + 2I\right)Y + I, \\ X =& I - A^*YA. \end{aligned} \tag{2.5}$$

The equation (2.5) yields

$$\begin{aligned} Y =& \left(\left((X - I)Y - 2I\right)X + 2I\right)Y + I \\ =& (XY - Y - 2I)XY + 2Y + I. \end{aligned}$$

Hence,

$$\begin{aligned} 0 =& \left((XY - I) - (Y + I)\right)XY + Y + I \\ =& (XY - I)XY - (Y + I)XY + (Y + I) \\ =& XY(XY - I) + (Y + I)(I - XY). \end{aligned}$$

Based on which, the following can be obtained

$$(Y + I - XY)(I - XY) = 0.$$

Since $I \leqslant Y$, $0 \leqslant I - X$, and $(I - X)Y = Y(I - X)$ are true, based on Theorem 5, $0 \leqslant (I - X)Y$ holds. We obtain $0 < I \leqslant (I - X)Y + I$. So, $Y + I - XY$ is nonsingular. We have $(Y + I - XY)^{-1}(Y + I - XY)(I - XY) = 0$. Thus, the result is $I - XY = 0$, i.e. $XY = I$. From (2.5) and what has obtained above, the equations $Y = X^{-1}$ and $X = I - A^*X^{-1}A$ are deduced. In addition, as $X_n \geqslant X_+$ is true , $X = X_+$ and $Y = X_+^{-1}$ are at hand easily.
If the matrix A is nonsingular and $X_n > 0$ for every n, then the above proof also shows that the sequence $\{Y_n\}$ is monotone increasing. This leads to the result that the sequence $\{X_n\}$ is monotone decreasing and bounded from below to the zero matrix. Therefore, $\lim_{n\to\infty} X_n = X$ exists. Since A is nonsingular, $Y_{n+1} = A^{-*}(I - X_{n+1})A^{-1}$ holds. Hence, $\lim_{n\to\infty} Y_n = Y$ also exists. Since $Y_0 = I$ and $\{Y_n\}$ are monotone increasing, $Y \geqslant I$ is obtained. Similarly, it can be concluded that $X = Y^{-1} > 0$, and finally $X = I - A^*X^{-1}A$ is available. As a result, the equation (1.1) has a positive definite solution. □

Lemma 5. *Suppose that* (1.1) *has a positive definite solution and* $\|A\| < \frac{1}{2}$. *In this case, the sequence* $\{Y_n\}$ *satisfy* $\|Y_n A\| < 1$, *for any* $n = 0, 1, \ldots$.

Proof. As $Y_0 = Y_1 = I$, it is clear that $\|Y_0 A\| = \|Y_1 A\| < \frac{1}{2} < 1$. For Y_2, we have

$$Y_2 = (((X_1 - I)Y_1 - 2I)\, X_1 + 2I)\, Y_1 + I = I + A^*A + (A^*A)^2.$$

Now, multiplying the upper equation by the matrix A from the right yields

$$Y_2 A = A + A^*A^2 + (A^*A)^2 A.$$

Thus,

$$\begin{aligned} \|Y_2 A\| = \|A + A^*A^2 + (A^*A)^2 A\| &\leq \|A\| + \|A^*A^2\| + \|(A^*A)^2 A\| \\ &\leq \frac{1}{2} + \frac{1}{8} + \frac{1}{32} = \frac{21}{32} < 1. \end{aligned}$$

So far, the inequality holds when $n = 0, 1, 2$. Therefore, we assume that the inequality is true for n, i.e. $\|Y_n A\| < 1$. Now, the inequality is proved for $n + 1$. It can be concluded that

$$\begin{aligned} Y_{n+1}A &= [(((X_n - I)Y_n - 2I)\, X_n + 2I)\, Y_n + I]\, A \\ &= [(((I - A^*Y_nA - I)Y_n - 2I)\, (I - A^*Y_nA) + 2I)\, Y_n + I]\, A \\ &= [((-A^*Y_nAY_n - 2I)(I - A^*Y_nA) + 2I)\, Y_n + I]\, A \\ &= [(-A^*Y_nAY_n(I - A^*Y_nA) - 2(I - A^*Y_nA) + 2I)\, Y_n + I]\, A \\ &= [(-A^*Y_nAY_n + A^*Y_nAY_nA^*Y_nA - 2I + 2A^*Y_nA + 2I)Y_n + I]\, A \\ &= [-A^*Y_nAY_k^2 + A^*Y_nAY_nA^*Y_nAY_n + 2A^*Y_nAY_n + I]A \\ &= -A^*Y_nAY_k^2A + A^*Y_nAY_nA^*Y_nAY_nA + 2A^*Y_nAY_nA + A \\ &= A^*Y_nA(-Y_n + Y_nA^*Y_nA + 2I)Y_nA + A \\ &= A^*Y_nA\,(-Y_n(I - A^*Y_nA) + 2I)\, Y_nA + A \\ &= A^*Y_nA(2I - Y_nX_n)Y_nA + A. \end{aligned}$$

According to Lemma 2, we have

$$(2I - Y_nX_n)Y_n = 2Y_n - Y_nX_nY_n \leqslant X_n^{-1} \leqslant X_+^{-1}.$$

Using Theorem 4, we note that $\frac{1}{2}I \leqslant X_+ \leqslant I$. Hence, $X_+^{-1} \leqslant 2I$ holds. It means $\|X_+^{-1}A\| < 1$. So, it results

$$\begin{aligned} \|Y_{n+1}A\| &\leq \|A^*Y_nA(2I - Y_nX_n)Y_nA + A\| \\ &\leq \|A^*\|\|Y_nA\|\|X_+^{-1}A\| + \|A\| \\ &\leq \|A^*\|\|Y_nA\| + \|A\| \\ &< \|A^*\| + \|A\| < 1. \end{aligned}$$

This completes the induction for $n + 1$ and thus the proof of the lemma. □

The following theorem shows the rate of convergence of the method (2.2).

Theorem 7. *If the matrix equation* (1.1) *has a positive definite solution, A is normal, and $\|A\| < \frac{1}{2}$ holds, then the sequences $\{X_n\}$ and $\{Y_n\}$ satisfy for any large n in*

$$\|Y_{n+1} - X_+^{-1}\| \leq \|A\|\|AX_+^{-1}\|\|Y_n - X_+^{-1}\|, \tag{2.6}$$

$$\|X_{n+1} - X_+\| \leq \|A\|^3\|AX_+^{-1}\|\|Y_n - X_+^{-1}\|, \tag{2.7}$$

and

$$\|X_{n+1} - X_+\| \leq \|A\|\|X_+^{-1}A\|\|X_n - X_+\|. \tag{2.8}$$

Proof. Based on Y_{n+1}, we have

$$\begin{aligned}
Y_{n+1} &= (((X_n - I)Y_n - 2I)\, X_n + 2I)\, Y_n + I \\
&= ((-A^*Y_nAY_n - 2I)(I - A^*Y_nA) + 2I)\, Y_n + I \\
&= A^*Y_nA(-Y_n + Y_nA^*Y_nA + 2I)Y_n + I \\
&= A^*(Y_n - X_+^{-1} + X_+^{-1})A\,(-Y_n(I - A^*Y_nA) + 2I)\, Y_n + I \\
&= A^*(Y_n - X_+^{-1} + X_+^{-1})A(2I - Y_nX_n)Y_n + I \\
&= A^*(Y_n - X_+^{-1})A(2I - Y_nX_n)Y_n + A^*X_+^{-1}A(2I - Y_nX_n)Y_n + I.
\end{aligned}$$

Therefore,

$$\begin{aligned}
X_+^{-1} - Y_{n+1} &= X_+^{-1} - A^*(Y_n - X_+^{-1})A(2I - Y_nX_n)Y_n \\
&\qquad - A^*X_+^{-1}A(2I - Y_nX_n)Y_n - I \\
&= A^*(X_+^{-1} - Y_n)A(2I - Y_nX_n)Y_n \\
&\qquad - (-X_+^{-1} + A^*X_+^{-1}A(2I - Y_nX_n)Y_n + I).
\end{aligned} \tag{2.9}$$

Hence, taking the norm from the above relation implies that

$$\begin{aligned}
\|X_+^{-1} - Y_{n+1}\| &\leq \|A^*(X_+^{-1} - Y_n)A(2I - Y_nX_n)Y_n\| \\
&\qquad + \| - X_+^{-1} + A^*X_+^{-1}A(2I - Y_nX_n)Y_n + I\|.
\end{aligned}$$

According to Lemma 2, we have

$$0 \leqslant (2I - Y_nX_n)Y_n = 2Y_n - Y_nX_nY_n \leqslant X_n^{-1} \leqslant X_+^{-1}. \tag{2.10}$$

Note that,

$$\begin{aligned}
&A^*X_+^{-1}A(2I - Y_nX_n)Y_n = (2I - Y_nX_n)Y_nA^*X_+^{-1}A, \\
&A^*X_+^{-1}AX_+^{-1} = X_+^{-1}A^*X_+^{-1}A, \\
&A^*(X_+^{-1} - Y_n)A(2I - Y_nX_n)Y_n = (2I - Y_nX_n)Y_nA^*(X_+^{-1} - Y_n)A, \\
&A^*(X_+^{-1} - Y_n)AX_+^{-1} = X_+^{-1}A^*(X_+^{-1} - Y_n)A,
\end{aligned}$$

hold. Based on Lemma 3 and inequality (2.10), since $0 \leqslant A^*X_+^{-1}A$ and $0 \leqslant A^*(X_+^{-1} - Y_n)A$ are true, we have

$$A^*X_+^{-1}A(2I - Y_nX_n)Y_n \leqslant A^*X_+^{-1}AX_+^{-1},$$

and

$$A^*(X_+^{-1} - Y_n)A(2I - Y_nX_n)Y_n \leqslant A^*(X_+^{-1} - Y_n)AX_+^{-1}.$$

It yields,

$$\| - X_+^{-1} + A^* X_+^{-1} A(2I - Y_n X_n) Y_n + I\| \le \| - X_+^{-1} + A^* X_+^{-1} A X_+^{-1} + I\|, \tag{2.11}$$

and

$$\|A^*(X_+^{-1} - Y_n) A(2I - Y_n X_n) Y_n\| \le \|A^*(X_+^{-1} - Y_n) A X_+^{-1}\|. \tag{2.12}$$

Notice that,

$$\begin{aligned} -X_+^{-1} + A^* X_+^{-1} A X_+^{-1} + I &= I + A^* X_+^{-1} A X_+^{-1} - X_+^{-1} \\ &= I + (A^* X_+^{-1} A - I) X_+^{-1} \\ &= I + (-X_+) X_+^{-1} = 0. \end{aligned} \tag{2.13}$$

Thus, using (2.11), (2.12), and (2.13), we have

$$\begin{aligned} \|Y_{n+1} - X_+^{-1}\| &\le \|A^*(X_+^{-1} - Y_n) A X_+^{-1}\| \\ &\qquad + \| - X_+^{-1} + A^* X_+^{-1} A X_+^{-1} + I\| \\ &\le \|A^*\| \|(X_+^{-1} - Y_n)\| \|A X_+^{-1}\| \\ &\le \|A^*\| \|A X_+^{-1}\| \|(X_+^{-1} - Y_n)\|. \end{aligned}$$

So, the inequality (2.6) is true. Based on the equation (1.1) for the solution X_+ and the sequence $\{X_n\}$ in the method (2.2),

$$X_+ = I - A^* X_+^{-1} A,$$

and

$$X_{n+1} = I - A^* Y_{n+1} A,$$

are obtained. As a result

$$\begin{aligned} X_{n+1} - X_+ &= (I - A^* Y_{n+1} A) - (I - A^* X_+^{-1} A) \\ &= I - A^* Y_{n+1} A - I + A^* X_+^{-1} A \\ &= A^*(X_+^{-1} - Y_{n+1}) A. \end{aligned}$$

Inequality (2.7) is obtained from the last equation above $X_{n+1} - X_+ = A^*(X_+^{-1} - Y_{n+1})A$. Therefore, taking the norm from this, we have the inequality

$$\begin{aligned} \|X_{n+1} - X_+\| &\le \|A\|^2 \|X_+^{-1} - Y_{n+1}\| \\ &\le \|A\|^3 \|A X_+^{-1}\| \|X_+^{-1} - Y_n\|. \end{aligned}$$

To prove the inequality (2.8), using the equation (2.9) yields

$$\begin{aligned} X_{n+1} - X_+ &= A^*(X_+^{-1} - Y_{n+1}) A \\ &= A^* \left(A^*(X_+^{-1} - Y_n) A(2I - Y_n X_n) Y_n \right. \\ &\qquad \left. + \left(X_+^{-1} - A^* X_+^{-1} A(2I - Y_n X_n) Y_n - I\right)\right) A \\ &= A^*(A^* X_+^{-1} A - A^* Y_n A)(2I - Y_n X_n) Y_n A \\ &\qquad + A^* \left(X_+^{-1} - A^* X_+^{-1} A(2I - Y_n X_n) Y_n - I\right) A \\ &= A^*(X_n - X_+)(2I - Y_n X_n) Y_n A \\ &\qquad - A^* \left(-X_+^{-1} + A^* X_+^{-1} A(2I - Y_n X_n) Y_n + I\right) A. \end{aligned}$$

Taking the norm of the above equation implies

$$\begin{aligned}\|X_{n+1} - X_+\| &= \|A^*(X_+^{-1} - Y_{n+1})A\| \\ &\leq \|A^*(X_n - X_+)(2I - Y_nX_n)Y_nA\| \\ &\quad + \|A^*\left(-X_+^{-1} + A^*X_+^{-1}A(2I - Y_nX_n)Y_n + I\right)A\|.\end{aligned}$$

Therefore, using (2.11), (2.12), and (2.13), we have

$$\begin{aligned}\|X_{n+1} - X_+\| &\leq \|A^*(X_n - X_+)X_+^{-1}A\| \\ &\quad + \|A^*\left(-X_+^{-1} + A^*X_+^{-1}AX_+^{-1} + I\right)A\| \\ &\leq \|A^*\|\|X_n - X_+\|\|X_+^{-1}A\| \\ &= \|A^*\|\|X_+^{-1}A\|\|X_n - X_+\|.\end{aligned}$$

Thus, the inequality (2.8) is established. □

It can be seen that the power of $\|A\|$ in the asymptotic convergence coefficients of the relations (2.6), (2.7) and (2.8) related to Theorem 7 are one unit higher than the similar cases in [3]. This feature makes the method (2.2) work faster.
It is clear that since the sequences $\{X_n\}$ and $\{Y_n\}$ converge to X_+ and X_+^{-1}, respectively, the relation $\|I - X_nY_n\| < \epsilon$ is obtained for all large enough ns.

Theorem 8. *If the equation* (1.1) *has one solution, then after n iterations we have from the method* (2.2)

$$\|X_n + A^*X_n^{-1}A - I\| \leq \epsilon\|A\|^2\|X_+^{-1}\|.$$

Proof. As

$$\begin{aligned}X_n + A^*X_n^{-1}A - I &= X_n + A^*X_n^{-1}A - (X_{n+1} + A^*Y_{n+1}A) \\ &= X_n - X_{n+1} + A^*(X_n^{-1} - Y_{n+1})A \\ &= A^*(Y_{n+1} - Y_n)A + A^*(X_n^{-1} - Y_{n+1})A \\ &= A^*(Y_{n+1} - Y_n + X_n^{-1} - Y_{n+1})A \\ &= A^*(X_n^{-1} - Y_n)A \\ &= A^*X_n^{-1}(I - X_nY_n)A.\end{aligned}$$

As a result, we take norms on both sides

$$\begin{aligned}\|X_n + A^*X_n^{-1}A - I\| &\leq \|A\|^2\|X_n^{-1}\|\|I - X_nY_n\| \\ &\leq \|A\|^2\|X_+^{-1}\|\|I - X_nY_n\| \\ &\leq \epsilon\|A\|^2\|X_+^{-1}\|.\end{aligned}$$

□

The value of ϵ used in Theorem 8 shows the effect of the first part of the method (2.2) on the norm of residual value of the maximal solution of $\|X_n + A^*X_n^{-1}A - I\|$ in the nth iteration. In Section 3, we use the norm of the residual value in the stop criterion of the methods, and we display the value of $\|I - X_nY_n\|$ obtained from the methods in examples.

3 Numerical results

In this section, some numerical examples are given to illustrate the efficiency of the method (2.2), in which the maximal positive definite solution for equations containing matrices A of different sizes is calculated. The calculations of this part have been done by MATLAB R2015b program on PC with specifications (AMD, E2-1800, APU, 1.70 GHz). The method (2.2) (Sh-Es) is compared with the methods (1.5) (Gu-La), (1.6) (El.S-Al.D) and (1.8) (Er-Sa-Es). For the following examples, we use the stop criterion

$$\|X_n + A^* X_n^{-1} A - I\| < 10^{-16}.$$

In each example, we report the number of iterations (Iter), the residual value of the response (Res), the elapsed execution time (CPU), and the total number of matrix-matrix multiplications (TMM) required for each method.

Although convergence of method (2.2) is proven for normal matrix A, numerical results indicate convergence in other cases as well. The matrix A is normal in the first two examples given.

Example 1. *Equation* (1.1) *can be concurrent with the matrix*

$$A = \begin{bmatrix} 0.21708 & 0.21708 & 0.21708 \\ 0.26587 & -0.26587 & 0 \\ 0.1535 & 0.1535 & -0.307 \end{bmatrix}.$$

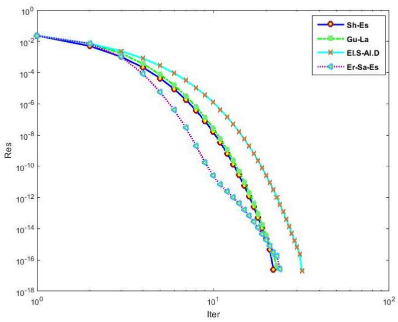
For this matrix, the spectral norm is equal to $\|A\| = 0.376$. *The maximal solution* X_+ *is:*

$$X_+ = \begin{bmatrix} 0.82958 & -1.9098 \times 10^{-6} & 4.1076 \times 10^{-7} \\ -1.9098 \times 10^{-6} & 0.82958 & 9.5842 \times 10^{-7} \\ 4.1076 \times 10^{-7} & 9.5842 \times 10^{-7} & 0.82958 \end{bmatrix}.$$

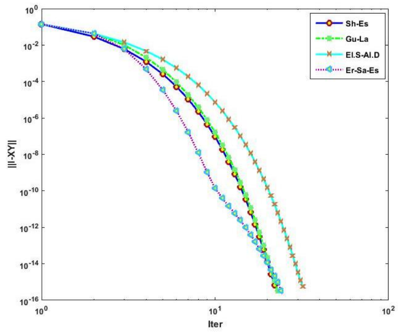
The results of this example are shown in Table 1 and Figure 1.

Table 1: The results of CPU, TMM, Iter, and Res related to Example 1

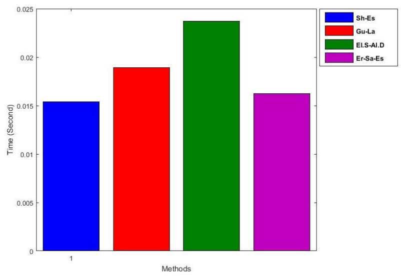
Method	CPU	TMM	Iter	Res
Sh-Es	$1.5408e-2$	88	22	$2.4037e-17$
Gu-La	$1.8925e-2$	92	23	$3.9619e-17$
El.S-Al.D	$2.3715e-2$	96	32	$1.9626e-17$
Er-Sa-Es	$1.6215e-2$	96	24	$2.6281e-17$



(a) $Res = \|X_n + A^* X_n^{-1} A - I\|$



(b) $\|I - X_n Y_n\|$



(c) The elapsed execution time of methods

Figure 1: The results of (a) Res, (b) $\|I - X_n Y_n\|$, and (c) CPU related to Example 1

Example 2. *Equation* (1.1) *can be shown with the matrix*

$$A = \frac{1}{100}\begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.3 & 0.6 & 0.15 \\ 0.4 & 0.15 & 0.6 \end{bmatrix}.$$

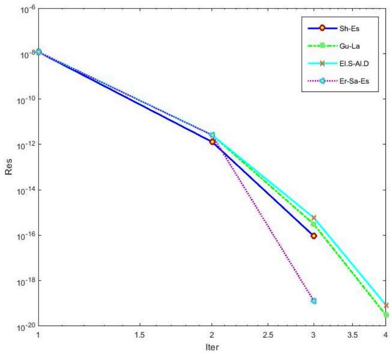
For this matrix, the spectral norm is equal to $\|A\| = 0.010434$*. The maximal solution* X_+ *is:*

$$X_+ = \begin{bmatrix} 0.99997 & -3.0003 \times 10^{-5} & -3.6504 \times 10^{-5} \\ -3.0003 \times 10^{-5} & 0.99995 & -3.0004 \times 10^{-5} \\ -3.6504 \times 10^{-5} & -3.0004 \times 10^{-5} & 0.99995 \end{bmatrix}.$$

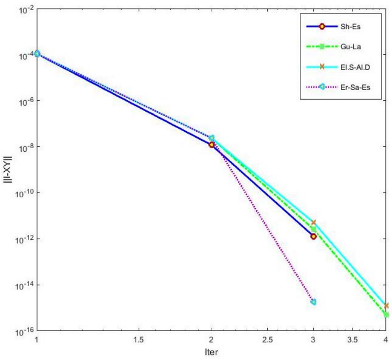
The results of this example are shown in Table 2 and Figure 2.

Table 2: The results of CPU, TMM, Iter, and Res related to Example 2

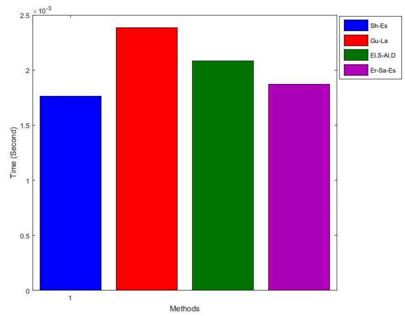
Method	CPU	TMM	Iter	Res
Sh-Es	$1.7617e-3$	12	3	$9.24e-17$
Gu-La	$2.3836e-3$	16	4	$3.0188e-20$
El.S-Al.D	$2.0857e-3$	12	4	$8.4025e-20$
Er-Sa-Es	$1.8707e-3$	12	3	$1.2817e-19$



(a) $Res = \|X_n + A^* X_n^{-1} A - I\|$



(b) $\|I - X_n Y_n\|$



(c) The elapsed execution time of methods

Figure 2: The results of (a) Res, (b) $\|I - X_n Y_n\|$, and (c) CPU related to Example 2

Example 3. *Equation* (1.1) *is considered with the matrix*

$$A = \begin{bmatrix} 0.1 & -0.15 & -0.2598076 \\ 0.15 & 0.2125 & -0.0649519 \\ 0.2598076 & 0.0649519 & 0.1375 \end{bmatrix}.$$

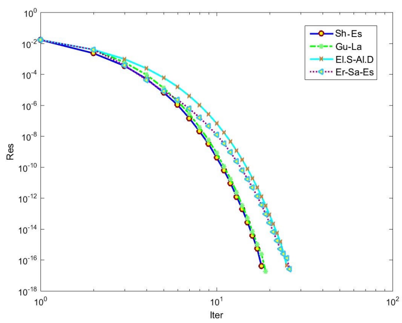
For this matrix, the spectral norm is equal to $\|A\| = 0.36512$. *The maximal solution* X_+ *is:*

$$X_+ = \begin{bmatrix} 0.88378 & -0.03948 & 0.0021512 \\ -0.03948 & 0.92245 & -0.038111 \\ 0.0021512 & -0.038111 & 0.89691 \end{bmatrix}.$$

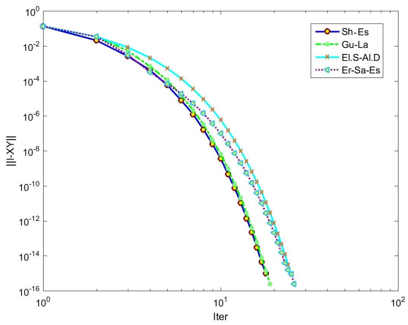
The results of this example are shown in Table 3 and Figure 3.

Table 3: The results of CPU, TMM, Iter, and Res related to Example 3

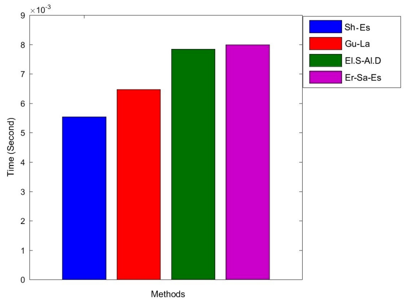
Method	CPU	TMM	Iter	Res
Sh-Es	$5.5455e-03$	72	18	$4.0103e-17$
Gu-La	$6.4765e-03$	76	19	$1.8359e-17$
El.S-Al.D	$7.8441e-03$	75	25	$4.4213e-17$
Er-Sa-Es	$8.0004e-03$	104	26	$2.6336e-17$



(a) $Res = \|X_n + A^* X_n^{-1} A - I\|$



(b) $\|I - X_n Y_n\|$



(c) The elapsed execution time of methods

Figure 3: The results of (a) Res, (b) $\|I - X_n Y_n\|$, and (c) CPU related to Example 3

The matrices A used in the examples 4, 5, 6, and 7 are generated based on the decomposition (1.2). In this case, first a random matrix with size $2m \times m$ is generated and after orthonormalization, the matrix $\binom{W}{Z}$ is obtained. Then the matrix $A = W^* Z$ is calculated, provided that W is nonsingular.

Example 4. *Equation* (1.1) *is considered with a square matrix*

$$A = \begin{bmatrix} -0.17733 & 0.016444 & -0.18063 \\ -0.13579 & 0.0026673 & -0.2739 \\ 0.045896 & -0.093311 & 0.28681 \end{bmatrix}.$$

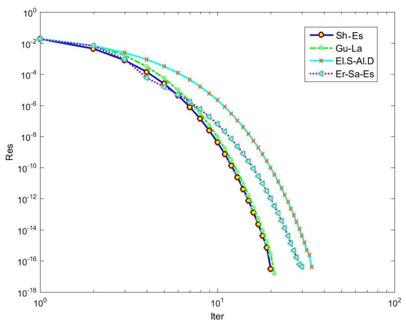
For this matrix, the spectral norm is equal to $\|A\| = 0.48232$. *The maximal solution* X_+ *is:*

$$X_+ = \begin{bmatrix} 0.94693 & 0.0075908 & -0.083593 \\ 0.0075908 & 0.98887 & 0.0036979 \\ -0.083593 & 0.036979 & 0.78911 \end{bmatrix}.$$

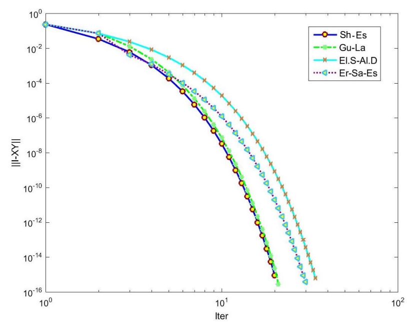
The results of this example are shown in Table 4 and Figure 4.

Table 4: The results of CPU, TMM, Iter, and Res related to Example 4

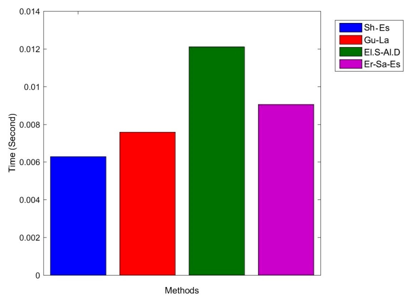
Method	CPU	TMM	Iter	Res
Sh-Es	$6.2884e-03$	80	20	$3.1306e-17$
Gu-La	$7.5907e-03$	84	21	$1.5749e-17$
El.S-Al.D	$1.2121e-02$	102	34	$3.9412e-17$
Er-Sa-Es	$9.0625e-03$	120	30	$4.2528e-17$



(a) $Res = \|X_n + A^* X_n^{-1} A - I\|$



(b) $\|I - X_n Y_n\|$



(c) The elapsed execution time of methods

Figure 4: The results of (a) Res, (b) $\|I - X_n Y_n\|$, and (c) CPU related to Example 4

Example 5. *Equation* (1.1) *is considered with a square matrix*

$$A = \begin{bmatrix} 0.14419 & 0.34069 & 0.10457 & -0.0014395 \\ 0.06218 & 0.12662 & -0.16326 & 0.22659 \\ 0.33037 & -0.016956 & 0.13182 & -0.1254 \\ 0.16004 & 0.040496 & -0.021911 & 0.16674 \end{bmatrix}.$$

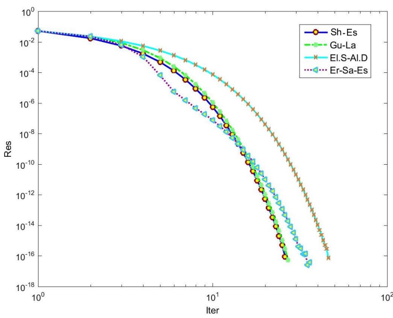
For this matrix, the spectral norm is equal to $\|A\| = 0.46657$*. The maximal solution* X_+ *is:*

$$X_+ = \begin{bmatrix} 0.8202 & -0.086328 & -0.046208 & -0.0085214 \\ -0.086328 & 0.82252 & -0.011205 & -0.056019 \\ -0.046208 & -0.011205 & 0.93679 & 0.069081 \\ -0.0085214 & -0.056019 & 0.069081 & 0.87923 \end{bmatrix}.$$

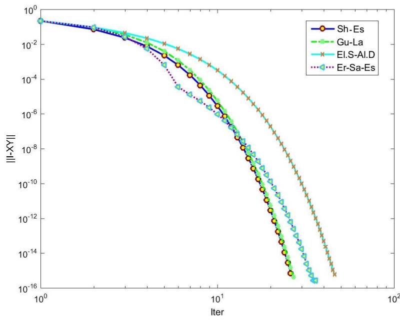
The results of this example are shown in Table 5 and Figure 5.

Table 5: The results of CPU, TMM, Iter, and Res related to Example 5

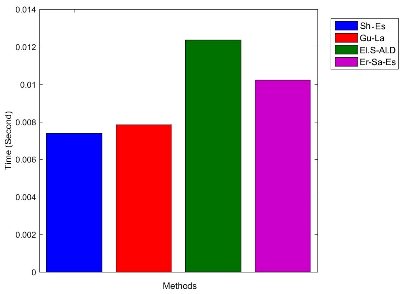
Method	CPU	TMM	Iter	Res
Sh-Es	$7.3987e-03$	104	26	$9.1116e-17$
Gu-La	$7.8520e-03$	108	27	$5.4920e-17$
El.S-Al.D	$1.2374e-02$	138	46	$7.6262e-17$
Er-Sa-Es	$1.0248e-02$	144	36	$3.7601e-17$



(a) $Res = \|X_n + A^* X_n^{-1} A - I\|$



(b) $\|I - X_n Y_n\|$



(c) The elapsed execution time of methods

Figure 5: The results of (a) Res, (b) $\|I - X_n Y_n\|$, and (c) CPU related to Example 5

Example 6. *Equation* (1.1) *is considered with a square matrix*

$$A = \begin{bmatrix} 0.03795 & -0.017676 & 0.12669 & 0.16814 & 0.083079 \\ -0.19121 & -0.21715 & 0.074911 & -0.017446 & -0.080205 \\ -0.058773 & 0.093033 & -0.18672 & 0.029281 & -0.012489 \\ 0.047295 & -0.18493 & -0.083502 & -0.24249 & 0.055405 \\ -0.17478 & 0.088897 & 0.0073937 & -0.13583 & -0.075462 \end{bmatrix}.$$

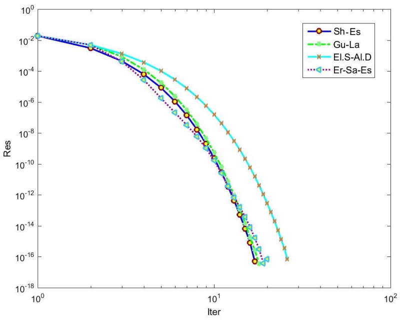
For this matrix, the spectral norm is equal to $\|A\| = 0.38827$. *The maximal solution* X_+ *is:*

$$X_+ = \begin{bmatrix} 0.92221 & -0.017102 & 0.0057636 & -0.021915 & -0.03583 \\ -0.017102 & 0.88631 & 0.022238 & -0.043605 & 0.0013055 \\ 0.0057636 & 0.022238 & 0.92908 & -0.037207 & -0.00094962 \\ -0.0021915 & -0.043605 & -0.037207 & 0.88592 & -0.011469 \\ -0.03583 & 0.0013055 & -0.00094962 & -0.011469 & 0.97698 \end{bmatrix}.$$

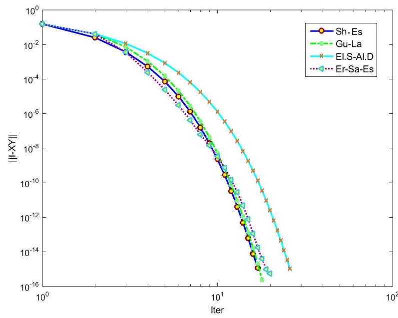
The results of this example are shown in Table 6 and Figure 6.

Table 6: The results of CPU, TMM, Iter, and Res related to Example 6

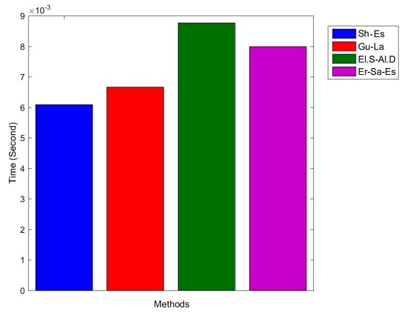
Method	CPU	TMM	Iter	Res
Sh-Es	$6.0928e-03$	68	17	$5.0815e-17$
Gu-La	$6.6716e-03$	72	18	$3.5276e-17$
El.S-Al.D	$8.7728e-03$	78	26	$7.2954e-17$
Er-Sa-Es	$7.9990e-03$	80	20	$7.2293e-17$



(a) $Res = \|X_n + A^* X_n^{-1} A - I\|$



(b) $\|I - X_n Y_n\|$



(c) The elapsed execution time of methods

Figure 6: The results of (a) Res, (b) $\|I - X_n Y_n\|$, and (c) CPU related to Example 6

Example 7. *Equation* (1.1) *is considered with a square matrix*

$$A = \begin{bmatrix} -0.095174 & 0.1434 & -0.15439 & 0.12221 & -0.062981 & -0.25131 \\ 0.13601 & -0.18154 & -0.1413 & 0.039451 & 0.040104 & -0.15815 \\ -0.07581 & -0.17715 & -0.21364 & 0.16425 & 0.12705 & 0.010581 \\ -0.0061382 & 0.065756 & -0.0094587 & -0.092901 & 0.28762 & 0.19148 \\ 0.0082114 & -0.1428 & -0.0050494 & -0.16838 & -0.20781 & 0.099261 \\ -0.081531 & 0.028218 & -0.11962 & 0.024314 & 0.18251 & 0.14977 \end{bmatrix}.$$

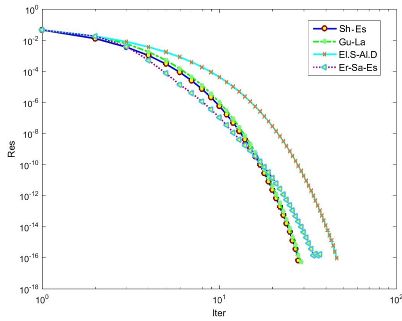
For this matrix, the spectral norm is equal to $\|A\| = 0.49804$. *The maximal solution* X_+ *is:*

$$X_+ = \begin{bmatrix} 0.95352 & 0.031685 & -0.025565 & 0.027123 & 0.015738 & 0.010516 \\ 0.031685 & 0.86995 & -0.054046 & 0.0046913 & -0.012424 & 0.0050747 \\ -0.025565 & -0.054046 & 0.8734 & 0.078287 & 0.050582 & -0.047582 \\ 0.027123 & 0.0046913 & 0.078287 & 0.89223 & -0.034766 & 0.084907 \\ 0.015738 & -0.012424 & 0.050582 & -0.034766 & 0.81112 & -0.069393 \\ 0.010516 & 0.0050747 & -0.047582 & 0.084907 & -0.069393 & 0.82277 \end{bmatrix}.$$

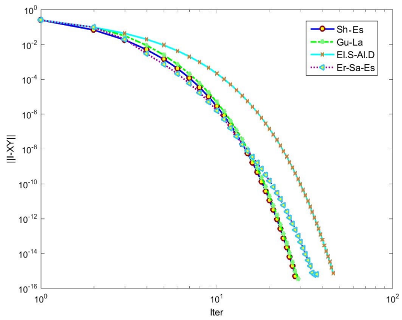
The results of this example are shown in Table 7 and Figure 7.

Table 7: The results of CPU, TMM, Iter, and Res related to Example 7

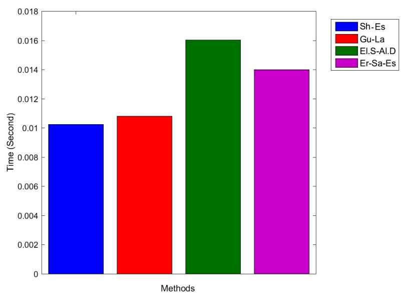
Method	CPU	TMM	Iter	Res
Sh-Es	$1.0249e-02$	112	28	$6.5666e-17$
Gu-La	$1.0814e-02$	116	29	$5.4278e-17$
El.S-Al.D	$1.6030e-02$	138	46	$9.6422e-17$
Er-Sa-Es	$1.4003e-02$	148	37	$1.6926e-16$



(a) $Res = \|X_n + A^* X_n^{-1} A - I\|$



(b) $\|I - X_n Y_n\|$



(c) The elapsed execution time of methods

Figure 7: The results of (a) Res, (b) $\|I - X_n Y_n\|$, and (c) CPU related to Example 7

Example 8. *Equation* (1.1) *is considered with ten randomly generated square matrices in sizes* 50×50 *and* 100×100 *with the average spectral norms* $\|A_{50\times 50}\| = 0.34109$ *and* $\|A_{100\times 100}\| = 0.47887$*, respectively. In this example, we use the stop criterion with the upper bound equal to* 10^{-10}*. The average results are reported in Table 8.*

Table 8: The results of CPU, TMM, Iter, and Res related to Example 8

50×50				
Method	CPU	TMM	Iter	Res
Sh-Es	$1.8674e-1$	28	7	$1.6651e-11$
Gu-La	$2.3584e-1$	30	7.5	$4.9412e-11$
El.S-Al.D	$2.9038e-1$	30.6	10.2	$2.9396e-11$
Er-Sa-Es	$3.9931e-1$	38.4	9.6	$7.0052e-11$

100×100				
Method	CPU	TMM	Iter	Res
Sh-Es	1.3211	40	10	$1.5709e-11$
Gu-La	1.4394	42.4	10.6	$4.3277e-11$
El.S-Al.D	2.1907	50.4	16.8	$9.1375e-11$
Er-Sa-Es	2.144	63.2	15.8	$5.8272e-11$

Based on the results of the examples, it can be seen that the method (2.2) requires less or equal number of iterations and the number of matrix-matrix multiplications than other methods to find the maximal solution. Also, the method (2.2) has the shortest execution time due to the number of iterations and the total number of matrix-matrix multiplications.

4 Conclusion

In this paper, we considered the nonlinear matrix equation (1.1) and proposed a new iterative inverse-free method to solve it. We obtained the convergence conditions of the method (2.2) and also found out that if the equation is solvable, one solution can be calculated numerically by the iterative method. Note that the method (2.2) involves inverse matrix approximation and matrix-matrix multiplications. In addition, the results of the given examples show that the method (2.2) performs faster than other methods and requires the least number of total matrix-matrix multiplications. The iterative method of generating the sequence $\{Y_n\}$ from the method (2.2) can be used independently as a new iterative method to calculate the inverse square matrix.

Declarations

- **Conflict of interest** The authors declare that they have no conflict of interest.
- **Funding** Not applicable.
- **Informed Consent Statement** The authors notify Consent Statement.
- **Availability of data and material** Data sharing does not apply to this article.
- **Authors' contributions** This is a joint work of both authors.

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