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Analytical solutions of Burgers hierarchy equations expressed in terms of generalized hypergeometric functions

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Abstract: The hierarchy of Burgers equations is considered. Using self-similar variables the hierarchy of ordinary differential equations is derived. Analytical solutions for ordinary differential equations, expressed in terms of a generalized hypergeometric function, are found. The Cauchy problem for a hierarchy of ordinary differential equations is considered. A relationship is found between solutions expressed in terms of a hypergeometric function and generalized Hermite polynomials. Direct transformation methods and mathematical induction were used in this work. The analytical solutions can provide benchmarks for validating numerical algorithms and assessing the convergence of computational schemes.

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1 Transforming hierarchy through group analysis

Solution of partial differential equations is one of the problems in applied mathematics. These equations describe a wide range of physical phenomena. One of the family of equations is the Burgers hierarchy. It can be written as follows [1]:

$$u_t + \frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x} + u\right)^k u\right] = 0, \qquad (k = 1, 2 \ldots) \tag{1}$$

Particular solutions of the linearized second-order equation of this hierarchy are expressed in terms of Hermite polynomials. They are used in various fields of natural sciences, such as physics, where they generate the eigenstates of a quantum harmonic oscillator [2]. They are also used in some cases of the heat conduction equation [3].

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In the paper [4], the generalized Hermite polynomials has been introduced. These polynomials express as family of particular solutions of higher-order hierarchical equations

$$P_n^{(k)} = \frac{\partial^n}{\partial t^n} exp\left\{zt - \frac{t^{k+1}}{2(k+1)}\right\}\Bigg|_{t=0}. \tag{2}$$

The Burgers hierarchy equations (1) are commonly used to describe physical processes. The most well-known equation is the Burgers equation presented about in 1948 [5] for modeling turbulence theory:

$$u_t + \frac{1}{2}u_{xx} + uu_x = 0. \tag{3}$$

In the second half of the 20th century, a third-order equation with $k = 2$ given, which called as the Sharma-Tasso-Olver equation

$$u_t + \frac{1}{2}u_{xxx} + \frac{3}{2}u_x^2 + \frac{3}{2}uu_{xx} + \frac{3}{2}u^2u_x = 0. \tag{4}$$

It describes the dynamic propagation of nonlinear dispersion waves in various heterogeneous environments [6, 7, 8].

1.1 Some transformations

Various methods are used to find solutions to partial differential equations, such as the method of group transformations. This method is used in this work. An operator is found with which the equation is reduced to an ordinary differential equation. The operator looks like this:

$$X = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u}. \tag{5}$$

This operator yields transformations for the variables x and t of the equation and the function $y(x, t)$:

$$z = x[(k+1)t]^{-\frac{1}{k+1}}, \tag{6}$$

$$u(x,t) = [(k+1)t]^{-\frac{1}{k+1}}F(z). \tag{7}$$

The substitutions (6) and (7) are substituted into the Burgers hierarchy equations (1) and they are integrated, then a differential equation in one variable with an arbitrary constant n is obtained. Next, n takes integer values:

$$\left[\left(\frac{d}{dz} + F\right)^k F\right] - 2zF + 2n = 0, \qquad (k = 1, 2, \ldots), \qquad n \in \mathbb{Z} \tag{8}$$

The transformations $u(x,t) = [(k+1)t]^{-\frac{1}{k+1}}F(z)$ and $z = x[(k+1)t]^{-\frac{1}{k+1}}$ are called stretching transformations.

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1.2 Cole-Hopf transformations

The equation (8) is invariant with respect to the transformation group and can be linearized using the Cole-Hopf transformation [9]:

$$F(x,t) = \partial_z \ln y. \tag{9}$$

After differentiation, the original equation is written as a linear ordinary differential equation in the following form:

$$y_{k+1,z} - 2zy_z + 2ny = 0, \qquad y_{k+1,z} = \frac{d^{k+1}y}{dz^{k+1}}, \qquad (k = 1, 2\ldots) \tag{10}$$

When $k = 1$, we obtain a second-order equation:

$$y_{zz} - 2zy_z + 2ny = 0. \tag{11}$$

2 ODE solution from the Burgers hierarchy

2.1 General solution of ODE from Burgers hierarchy

The solution to Eq. (11) is written in terms of hypergeometric functions [10]. This is solutions of a second-order ODE, which in general form is:

$$y'' + A(z)y' + B(z)y = 0, \tag{12}$$

$$\frac{A'}{2} + \frac{A^2}{4} - B \quad \text{– rational function,} \tag{13}$$

where $A(z) = -2z$, $B(z) = 2n$, $\frac{A'}{2} + \frac{A^2}{4} - B = z^2 - 2n - 1$. Equation (12) is obtained using equivalent fractional-linear transformations from an equation of the type pFq, where $p = 1$, $q = 1$. The function $y(z)$ is decomposed as the product of an arbitrary function and a polynomial with arbitrary, not necessarily natural, exponents. In general form, the transformations are written as:

$$\begin{aligned} z &\to \frac{\alpha t^k + \beta}{\gamma t^k + \delta}, \\ y(z) &\to P(t)u(t), \end{aligned} \tag{14}$$

where $\alpha, \beta, \gamma, \delta$ – real numbers, $P(t)$ – polynomial.
Hypergeometric equation looks like

$$zy'' + (c - z)y' - ay = 0, \tag{15}$$

where a, c are parameters.
The solution is expressed in terms of hypergeometric functions or their reduced case—Kummer functions $M(a, b, z)$ and $U(a, b, z)$. The equivalent transformations (14) are as follows:

$$\begin{aligned} z &\to t^2, \\ y &\to \frac{1}{t}u(t). \end{aligned} \tag{16}$$

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The values of parameters a and c are uniquely determined by selection. After substituting them into the equations for hypergeometric functions, the result repeats (10) with the value $k = 1$:

$$\begin{aligned} a &= \frac{1}{2} - \frac{n}{2}, \\ c &= \frac{3}{2}. \end{aligned} \tag{17}$$

The transformed equation coincides with the original one:

$$u''(t) - 2tu'(t) + 2nu(t) = 0. \tag{18}$$

To simplify the recording of new transformations, the variables are replaced back to $t \to z$, $u(y) \to y(z)$. Since the values of all parameters and the general form of the solution formula are known, the answer for the first ODE from the hierarchy is recorded:

$$y(z) = C_1 \cdot M\left(\frac{1}{2} - \frac{n}{2}, \frac{3}{2}, z^2\right) \cdot z + C_2 \cdot U\left(\frac{1}{2} - \frac{n}{2}, \frac{3}{2}, z^2\right) \cdot z. \tag{19}$$

The solution is expressed in terms of hypergeometric functions. A lemma is proved concerning the connection between the ODE from the Burgers hierarchy and the general form of the equation for hypergeometric functions, the solutions of which are known. From the equation of the form pFq with undefined parameters p and q, solutions for arbitrary order k are obtained. From these, a general solution is written down. An example of the equation itself is taken from Gelfand's article [11], $p = 1$, $q \in \mathbb{N}$ coincides with order k:

$$\frac{d}{dz} \prod_{j=1}^{k} \left(z\frac{d}{dz} + b_j - 1\right) y(z) - \left(z\frac{d}{dz} + a\right) y(z) = 0. \tag{20}$$

The solution to the equation is expressed as the product of an arbitrary function depending on z and the hypergeometric function pFq:

$$y(z) = g(z)F(a, [b_1, \ldots, b_k], f(z)). \tag{21}$$

The substitution that links the ODE and equation (20): $y(z) = \frac{u(t)}{g(t)}$, $g(t) \neq 0$, $z = f(t)$. Without loss of generality, the variables are renamed: $t \to x$, $u \to y$.
A lemma is recorded that proves the existence of an equivalent transformation between the equation for hypergeometric functions and the ODEs obtained from the Burgers hierarchy. The linear independence of $k + 1$ solutions to these equations is also shown.

Lemma 1. *For any $k \in \mathbf{N}$, equation (20) can be reduced to equation (10) using the transformations $z = \frac{2t^{k+1}}{(k+1)^k}$ and $y(z) = \frac{u(t)}{t^s}$, where the coefficients are as follows: $a = \frac{s}{k+1} - \frac{n}{k+1}$, $b_1 \ldots b_k, b_{k+1} = \frac{s+1}{k+1}, \ldots, \frac{k+s}{k+1}, \frac{k+s+1}{k+1}$, the solution is represented as the sum of hypergeometric functions (21).*

Proof. The proof is carried out using mathematical induction.
Base:
When $k = 2$, the substitution $z = \frac{2t^3}{9}$, $\frac{d}{dz} = \frac{3}{2t^2}\frac{d}{dt}$ follows from the condition of the lemma and is substituted into equation (20):

$$\frac{3}{2t^2}\frac{d}{dt}\left(\frac{t}{3}\frac{d}{dt} + b_1 - 1\right)\left(\frac{t}{3}\frac{d}{dt} + b_2 - 1\right)\frac{u(t)}{t^s} - \left(\frac{t}{3}\frac{d}{dt} + a\right)\frac{u(t)}{t^s} = 0. \tag{22}$$

The parameters are similar to the condition of the lemma: $a = \frac{s}{3} - \frac{n}{3}$ and $b_1, b_2, b_3 = \frac{s+1}{3}, \frac{s+2}{3}, \frac{s+3}{3}$. After substituting them into (22) and simplifying, the equation looks as follows:

$$u_{ttt}(t) - 2u_t(t)t + 2nu(t) = 0. \tag{23}$$

Induction step:
Let the condition of the lemma hold for arbitrary k. All parts of the substitution are taken from the condition of the lemma. The transformed differentiation operator is also written down:

$$\begin{aligned} z &= \frac{2t^{k+1}}{(k+1)^k}, \\ \frac{d}{dz} &= \frac{(k+1)^{k-1}}{2t^k}\frac{d}{dt}, \\ a &= \frac{s}{k+1} - \frac{n}{k+1}, \\ b_1, \dots, b_k, b_{k+1} &= \frac{s+1}{k+1}, \dots, \frac{k+s}{k+1}, \frac{k+s+1}{k+1}. \end{aligned} \tag{24}$$

After substituting the replacement into (20), the equation is transformed into:

$$\begin{aligned} &\frac{(k+1)^{k-1}}{2t^k}\frac{d}{dt}\left[\prod_{j=k-s+2}^{k+1}\left(\frac{t}{k+1}\frac{d}{dt} + \frac{j+s}{k+1} - 1\right)\right. \\ &\left.\prod_{j=1}^{k-s}\left(\frac{t}{k+1}\frac{d}{dt} + \frac{j+s}{k+1} - 1\right)\frac{u(t)}{t^s}\right] - \\ &-\left(\frac{t}{k+1}\frac{d}{dt} + \left(\frac{s}{k+1} - \frac{n}{k+1}\right)\right)\frac{u(t)}{t^s} = 0. \end{aligned} \tag{25}$$

The proof shows that it reduces to the form (10) by induction. The operators cannot be expanded explicitly, since the degree of the operator is undefined. To do this, each term in equation (25) is considered separately and checked for $k+1$. First, the transformations are performed for the second term, since it contains only one differentiation operator:

$$\begin{aligned} &-\left(\frac{t}{k+1}\frac{d}{dt} + \left(\frac{s}{k+1} - \frac{n}{k+1}\right)\right)\frac{u(t)}{t^s} = \\ &= -\frac{t}{k+1}\left(u'(t)\frac{1}{t^s} - u(t)\frac{1}{t^{s+1}}\right) - \left(\frac{s}{k+1} - \frac{n}{k+1}\right)\frac{u(t)}{t^s} = \\ &= -\frac{2tu'(t)}{2(k+1)t^s} + \frac{2su(t)}{2(k+1)t^s} - \frac{2su(t)}{2(k+1)t^s} + \frac{2nu(t)}{2(k+1)t^s} = \frac{-2u'(t)t + 2nu(t)}{2(k+1)t^s}. \end{aligned} \tag{26}$$

For $k+1$, the transformed equation is similar to the case for k. It is obtained either by expanding the brackets or by multiplying both sides of equality (26) by $\frac{k+1}{k+2}$:

$$\begin{aligned} &-\left(\frac{t}{k+2}\frac{d}{dt} + \left(\frac{s}{k+2} - \frac{n}{k+2}\right)\right)\frac{u(t)}{t^s} = \\ &= -\frac{t}{k+2}\left(u'(t)\frac{1}{t^s} - u(t)\frac{1}{t^{s+1}}\right) - \left(\frac{s}{k+2} - \frac{n}{k+2}\right)\frac{u(t)}{t^s} = \\ &= -\frac{2tu'(t)}{2(k+2)t^s} + \frac{2su(t)}{2(k+2)t^s} - \frac{2su(t)}{2(k+2)t^s} + \frac{2nu(t)}{2(k+2)t^s} = \frac{-2u'(t)t + 2nu(t)}{2(k+2)t^s}. \end{aligned} \tag{27}$$

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The first term is considered separately, in which the number of multipliers depends on k. The new expression is obtained from the previous one by differentiation. First, the first term for k and what it equals are written down. Next, the components of the left side are reduced to a single product operator:

$$\frac{(k+1)^k}{2t^{k+1}}\prod_{j=1}^{k+1}\left(\frac{t}{k+1}\frac{d}{dt}+\frac{j+s}{k+1}-1\right)\frac{u(t)}{t^s}=\frac{u_{k+1,t}}{2(k+1)t^s}. \tag{28}$$

And what needs to be obtained is recorded, replacing k with $k+1$:

$$\frac{(k+2)^{k+1}}{2t^{k+2}}\prod_{j=1}^{k+2}\left(\frac{t}{k+2}\frac{d}{dt}+\frac{j+s}{k+2}-1\right)\frac{u(t)}{t^s}=\frac{u_{k+2,t}}{2(k+2)t^s}. \tag{29}$$

By differentiating and rearranging the terms on the right-hand side of equation (28) is obtained the right-hand side of (29)

$$\begin{aligned}\frac{k+1}{k+2}\frac{d}{dt}\left[\frac{(k+1)^k}{2t^{k+1}}\prod_{j=1}^{k+1}\left(\frac{t}{k+1}\frac{d}{dt}+\frac{j+s}{k+1}-1\right)\frac{u(t)}{t^s}\right]=\\=\frac{k+1}{k+2}\frac{d}{dt}\frac{u_{k+1,t}}{2(k+1)t^s}.\end{aligned} \tag{30}$$

Let us take the replacement

$$\frac{(k+1)^k}{2t^{k+1}}\prod_{j=1}^{k+1}\left(\frac{t}{k+1}\frac{d}{dt}+\frac{j+s}{k+1}-1\right)\frac{u(t)}{t^s}=J(t). \tag{31}$$

In (30), only $k+2$ derivatives remain on the right-hand side:

$$\frac{su_{k+1,t}}{2(k+2)t^{s+1}}+\frac{k+1}{k+2}\frac{d}{dt}J(t)=\frac{u_{k+2,t}}{2(k+2)t^s}. \tag{32}$$

An external differentiation operator acts on the left side of the entire function. Similar terms are reduced:

$$\begin{aligned}\frac{k+1}{k+2}\frac{s}{t}\cdot\frac{u_{k+1,t}}{2(k+1)t^s}-\frac{k+1}{k+2}\frac{s}{t^{s+1}}\frac{(k+1)^k}{2t^{k+1-s}}\prod_{j=1}^{k+1}\left(\frac{t}{k+1}\frac{d}{dt}+\frac{j+s}{k+1}-1\right)\frac{u(t)}{t^s}+\\+\frac{k+1}{k+2}\frac{1}{t^s}\frac{d}{dt}\frac{(k+1)^k}{2t^{k+1-s}}\prod_{j=1}^{k+1}\left(\frac{t}{k+1}\frac{d}{dt}+\frac{j+s}{k+1}-1\right)\frac{u(t)}{t^s}=\\=\frac{k+1}{k+2}\frac{s}{t}\cdot\frac{u_{k+1,t}}{2(k+1)t^s}-\frac{k+1}{k+2}\frac{s}{t}\cdot\frac{u_{k+1,t}}{2(k+1)t^s}+\\+\frac{k+1}{k+2}\frac{1}{t^s}\frac{d}{dt}\frac{(k+1)^k}{2t^{k+1-s}}\prod_{j=1}^{k+1}\left(\frac{t}{k+1}\frac{d}{dt}+\frac{j+s}{k+1}-1\right)\frac{u(t)}{t^s}=\\=\frac{k+1}{k+2}\frac{1}{t^s}\frac{d}{dt}\frac{(k+1)^k}{2t^{k+1-s}}\prod_{j=1}^{k+1}\left(\frac{t}{k+1}\frac{d}{dt}+\frac{j+s}{k+1}-1\right)\frac{u(t)}{t^s}=\frac{u_{k+2,t}}{2(k+2)t^s}.\end{aligned} \tag{33}$$

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After that, the denominator of the components is changed in all factors. To do this, $\frac{1}{(k+1)^{k+1}}$ is taken out of the parentheses:

$$\begin{gathered}\frac{1}{(k+2)}\frac{1}{t^s}\frac{d}{dt}\frac{(k+1)^{k+1}}{2t^{k+1-s}(k+1)^{k+1}}\left[\prod_{j=1}^{k+1}\left(t\frac{d}{dt}+j+s-(k+1)\right)\frac{u(t)}{t^s}\right]= \\ =\frac{1}{(k+2)}\frac{1}{t^s}\frac{d}{dt}\frac{1}{2t^{k+1-s}}\left[\prod_{j=1}^{k+1}\left(t\frac{d}{dt}+j+s-(k+1)\right)\frac{u(t)}{t^s}\right].\end{gathered} \tag{34}$$

The denominator in all multipliers must be equal to $k+2$. Therefore, the entire expression is divided and multiplied by $(k+2)^{k+1}$. Since this is a constant value, it is placed under the product operator and distributed among all multipliers:

$$\frac{(k+2)^{k+1}}{(k+2)}\frac{1}{t^s}\frac{d}{dt}\frac{1}{2t^{k+1-s}}\left[\prod_{j=1}^{k+1}\left(\frac{t}{k+2}\frac{d}{dt}+\frac{j+s+1}{k+2}-1\right)\frac{u(t)}{t^s}\right]. \tag{35}$$

Replacing the index at $i=j+1$:

$$\frac{(k+2)^{k}}{t^s}\frac{d}{dt}\frac{1}{2t^{k+1-s}}\left[\prod_{i=2}^{k+2}\left(\frac{t}{k+2}\frac{d}{dt}+\frac{i+s}{k+2}-1\right)\frac{u(t)}{t^s}\right]. \tag{36}$$

Next, the first operator in the formula is differentiated. If $k+1-s>0$, after differentiation, the terms are reduced to the form (29). If $k+1-s\leq 0$, either there are not enough of them, or the second term in the final expression has the opposite sign. This means that $s<k+1$. An ODE of order $k+1$ has exactly the same number of solutions. Formula (36) looks like this:

$$\begin{gathered}\frac{(k+2)^{k}}{t^s}\frac{s-k-1}{2t^{k+2-s}}\prod_{i=2}^{k+2}\left(\frac{t}{k+2}\frac{d}{dt}+\frac{i+s}{k+2}-1\right)\frac{u(t)}{t^s}+ \\ +\frac{(k+2)^{k+1}}{t^{k+2}}\left(\frac{t}{k+2}\frac{d}{dt}+\frac{s-k-1}{k+2}\right)\prod_{i=2}^{k+2}\left(\frac{t}{k+2}\frac{d}{dt}+\frac{i+s}{k+2}-1\right)\frac{u(t)}{t^s}= \\ =\frac{(k+2)^{k+1}}{t^{k+2}}\prod_{i=1}^{k+2}\left(\frac{t}{k+2}\frac{d}{dt}+\frac{i+s}{k+2}-1\right)\frac{u(t)}{t^s}=\frac{u_{k+2,t}}{2(k+2)t^s}.\end{gathered} \tag{37}$$

Formulas (33) and (29) coincide, which means that the equality is proven. Grouping the results of formulas (27) and (33) gives the following final equation:

$$\begin{gathered}\frac{u_{k+2,t}(t)}{2(k+2)t^s}+\frac{-2u'(t)t+2nu(t)}{2(k+2)t^s}=0, \\ u_{k+2,t}(t)-2u'(t)t+2nu(t)=0.\end{gathered} \tag{38}$$

The lemma is proven.

□

To prove linear independence, the resulting solution is written in explicit form as a hypergeometric function:

$${}_pF_q(a_1,\dots a_p,b_1,\dots b_q,z)=\sum_{i=0}^{\infty}\frac{(a_1)_i\cdot(a_2)_i\cdot\dots\cdot(a_p)_i}{(b_1)_i\cdot(b_2)_i\cdot\dots\cdot(b_q)_i}\frac{z^i}{i!}, \tag{39}$$

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where $(a)_i = a(a+1)\dots(a+i-1), i \geq 1$. Here, $p = 1$ and k hypergeometric functions with different parameters are considered. Formula (39) is transformed to the following form:

$$ {}_1F_k^{(s)}(a_1^{(s)}, b_1^{(s)} \dots b_k^{(s)}, z) = \sum_{i=0}^{\infty} \frac{(a_1^{(s)})_i}{(b_1^{(s)})_i \cdot (b_2^{(s)})_i \cdot \dots \cdot (b_k^{(s)})_i} \frac{z^i}{i!} = \sum_{i=0}^{\infty} A_i^{(s)} \frac{z^i}{i!}. \tag{40} $$

Let two values s equal t and r such that $0 \leq t < r \leq k$. In this case, the two solutions of the hypergeometric equation are:

$$ \begin{aligned} f_1 &= z^t \sum_{i=0}^{\infty} A_i^{(t)} \frac{z^i}{i!}, \\ f_2 &= z^r \sum_{i=0}^{\infty} A_i^{(r)} \frac{z^i}{i!}. \end{aligned} \tag{41} $$

All coefficients $A_i^{(s)}$ are nonzero. For any fixed n, the highest degree of f_2 is always greater than the highest degree of f_1. The coefficient at the highest degree is also always nonzero. Since it is possible to fix any n, this rule holds for the entire series. It turns out that the functions included in the solution of the equation are all linearly independent of each other.
The solution of ODE (10) for any value of k is represented as the sum of a series of hypergeometric functions

$$ y(z) = \sum_{i=0}^{k} C_i z^i {}_1F_k \left(\frac{i}{k+1} - \frac{n}{k+1}, \underbrace{b_1, \dots b_k}_{\text{k}}, \frac{2z^{k+1}}{(k+1)^k} \right) \tag{42} $$

with given parameters

$$ a = \frac{i}{k+1} - \frac{n}{k+1} $$
$$ b_1 \dots b_k = \begin{cases} \frac{1}{k+1} \cdots \frac{k}{k+1}, & i = 0 \\ \frac{i+1}{k+1}, \dots \frac{i+1+k}{k+1}, & i = 1 \dots k-1, b_j \neq \frac{k+1}{k+1} \\ \frac{k+2}{k+1} \cdots \frac{2k+1}{k+1}, & i = k. \end{cases} \tag{43} $$

This is true for any natural number k and any value of n.

2.2 Search for coefficients of the general solution corresponding to Hermite generalized polynomials

For $k = 1$, the particular solutions are the classical Hermite polynomials. For these, there is a well-known representation using hypergeometric functions for even and odd powers:

$$ \begin{aligned} H_{2m}(z) &= \frac{2^{2m}\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - m\right)} F\left(-m; \frac{1}{2}; z^2\right), \\ H_{2m+1}(z) &= \frac{2^{2m+2}\sqrt{\pi}}{\Gamma\left(-\frac{1}{2} - m\right)} F\left(-m; \frac{3}{2}; z^2\right). \end{aligned} \tag{44} $$

In general, the solution for any degree n is written using the reduced Hermite polynomials:

$$ y(z) = \frac{1 + (-1)^n}{2} \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - \frac{n}{2}\right)} F\left(-\frac{n}{2}; \frac{1}{2}; z^2\right) - \frac{1 - (-1)^n}{2} \frac{2\sqrt{\pi}}{\Gamma\left(-\frac{n}{2}\right)} F\left(\frac{1}{2} - \frac{n}{2}; \frac{3}{2}; z^2\right). \tag{45} $$

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Then the initial conditions are:
if n is even

$$y(0) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - \frac{n}{2}\right)}, \quad y'(0) = 0.$$

if n is odd

$$y(0) = 0, \quad y'(0) = -\frac{2\sqrt{\pi}}{\Gamma\left(-\frac{n}{2}\right)}.$$

The search for coefficients is extended to other k. The numerator $(a_1^{(s)})_i$, formed by the parameter $a_1^{(s)}$, has the form: $a_1^{(s)} \cdot (a_1^{(s)} + 1) \cdot \ldots \cdot (a_1^{(s)} + i - 1)$, where $a_1^{(s)} = \frac{s}{k+1} - \frac{n}{k+1}$ for a fixed order k and degree n. The value of the parameter s takes all integers from 0 to k, that is, all possible remainders from dividing the natural number n by $k+1$. It turns out that $a_1^{(s)}$ is equal to an integer, and a non-positive one at that, only when $s \equiv n \pmod{(k+1)}$. Then, from a certain value i equal to $i = \frac{n-s}{k+1} + 1$, all terms of the infinite series of the hypergeometric function are equal to 0, since $(a_1^{(s)})_i$ has a zero factor. However, if $s \not\equiv n \pmod{(k+1)}$, the series always remains an infinite sum. No zero factor is formed in the numerator, and the parameters $b_1^{(s)}, b_2^{(s)}, \ldots, b_k^{(s)}$ are positive numbers.
From the general solution, it follows that for each generalized Hermite polynomial of degree n, only one term from the sum for the complete solution is necessary, where $s \equiv n \pmod{(k+1)}$.
Formula (2) represents generalized Hermite polynomials. In the hypergeometric function, the coefficient of the highest degree is most often not equal to 1. Therefore, to normalize the polynomial, it is multiplied by the reciprocal of this coefficient. The coefficients in the general solution before the hypergeometric functions are the reciprocals of the coefficients of the highest powers. Therefore, it is necessary to write down the general form of these coefficients.
When $s \equiv n \pmod{(k+1)}$, all terms in formula (40) starting with $i = \frac{n-s}{k+1} + 1$ are equal to 0. Therefore, the last term in the sum has the number $i = \left[\frac{n}{k+1}\right]$. Then, the formula for the coefficient contains the multiplier $\left(\frac{2}{(k+1)^k}\right)^{\left[\frac{n}{k+1}\right]}$.
The coefficient also contains multipliers formed using the parameters of the hypergeometric function. In the numerator, this is $(a_1^{(s)})_i$, where $i = \left[\frac{n}{k+1}\right]$, $s \equiv n \pmod{(k+1)}$. For any n, using transformations: $(a_1^{(s)})_i = (-1)^{\left[\frac{n}{k+1}\right]} \cdot \left[\frac{n}{k+1}\right]!$, with corresponding s, i, n, and k.
It remains to determine the contribution to the coefficient of parameters $(b_1^{(s)})_i, (b_2^{(s)})_i, \ldots (b_k^{(s)})_i$, where the values of s, i, n and k are similar to the parameter a. The number of parameters b coincides with the order of the equation k. From the form of the hypergeometric function, it follows that each of them consists of $\left[\frac{n}{k+1}\right]$ multipliers. They all have the same denominator $k+1$. The numerator of the coefficient before the highest power contains the multiplier $(k+1)^{k\left[\frac{n}{k+1}\right]}$. The intermediate result is as follows:

$$\frac{2^{\left[\frac{n}{k+1}\right]} \cdot (-1)^{\left[\frac{n}{k+1}\right]} \cdot \left[\frac{n}{k+1}\right]! \cdot (k+1)^{k\left[\frac{n}{k+1}\right]}}{(k+1)^{k\left[\frac{n}{k+1}\right]}} = 2^{\left[\frac{n}{k+1}\right]} \cdot (-1)^{\left[\frac{n}{k+1}\right]} \cdot \left[\frac{n}{k+1}\right]!$$

The parameters $(b_1^{(s)})_i, (b_2^{(s)})_i, \ldots (b_k^{(s)})_i$ are formed using the same formula as the parameter $(a_1^{(s)})_i$. In this case, a factorial is formed in the denominator. It does not include all factors that are multiples of $k+1$. Also, since the smallest numerator of the parameters b is equal to $s+1$, it does not contain all factors up to and including s. The factorial that is written in the denominator

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of the coefficient: $\left((s+k+1)+(k+1)\left(\left[\frac{n}{k+1}\right]-1\right)\right)! = \left(s+(k+1)\left[\frac{n}{k+1}\right]\right)!$. The numerator contains factors that are excluded from this expression: $s!\cdot\left((k+1)+(k+1)\left(\left[\frac{n}{k+1}\right]-1\right)\right)!_{(k+1)} = s!\cdot\left((k+1)\left[\frac{n}{k+1}\right]\right)!_{(k+1)}$. After all transformations, the coefficient in the hypergeometric function before the highest degree:

$$\frac{2^{\left[\frac{n}{k+1}\right]}\cdot(-1)^{\left[\frac{n}{k+1}\right]}\cdot\left[\frac{n}{k+1}\right]!\cdot s!\cdot\left((k+1)\left[\frac{n}{k+1}\right]\right)!_{(k+1)}}{\left(s+(k+1)\left[\frac{n}{k+1}\right]\right)!}.$$

Since all of Hermite's generalized polynomials are reduced, for each of them, a nonzero coefficient with the number $s \equiv n \pmod{(k+1)}$ is recorded in the general solution. It is the inverse of the highest power of the hypergeometric function:

$$C_s = \frac{(-1)^{\left[\frac{n}{k+1}\right]}\cdot\left(s+(k+1)\left[\frac{n}{k+1}\right]\right)!}{2^{\left[\frac{n}{k+1}\right]}\cdot\left[\frac{n}{k+1}\right]!\cdot s!\cdot\left((k+1)\left[\frac{n}{k+1}\right]\right)!_{(k+1)}}. \tag{46}$$

Taking into account this coefficient (46), the general solution of the ODE obtained from the Burgers hierarchy will coincide with the particular solutions—the generalized Hermite polynomials.
For these solutions, we can write the general form of the Cauchy problem for the corresponding hierarchy of ODEs. Let's remember what the hierarchy itself looked like

$$\begin{gathered}
y_{k+1,z}-2zy_z+2ny=0, \qquad y_{k+1,z}=\frac{d^{k+1}y}{dz^{k+1}}, \qquad (k=1,2\dots) \\
for \quad s\neq n \pmod{(k+1)} \quad and \quad (s=0,1\dots k), \quad y^{(s)}=0, \\
for \quad s\equiv n \pmod{(k+1)}, \quad y^{(s)}=C_s\frac{s!\cdot(s-n)}{(k+1)\cdot b_1\cdot\ldots\cdot b_k}, \\
where \quad C_s=\frac{(-1)^{\left[\frac{n}{k+1}\right]}\cdot\left(s+(k+1)\left[\frac{n}{k+1}\right]\right)!}{2^{\left[\frac{n}{k+1}\right]}\cdot\left[\frac{n}{k+1}\right]!\cdot s!\cdot\left((k+1)\left[\frac{n}{k+1}\right]\right)!_{(k+1)}}, \\
b_1\dots b_k=\begin{cases}\frac{1}{k+1}\cdots\frac{k}{k+1}, & \text{if} \quad s=0\\ \frac{s+1}{k+1}\cdots\frac{s+1+k}{k+1}, & \text{if} \quad s=1\dots k-1, b_j\neq\frac{k+1}{k+1}\\ \frac{k+2}{k+1}\cdots\frac{2k+1}{k+1}, & \text{if} \quad s=k\end{cases}
\end{gathered} \tag{47}$$

The analytical solutions obtained for the Burgers hierarchy, expressed in terms of generalized hypergeometric functions, are of significant value to computational mathematics and numerical analysis.
First, they can serve as benchmark tests for verifying and validating numerical algorithms designed to solve nonlinear partial differential equations. The availability of an exact closed-form solution allows one to quantitatively estimate the order of convergence of finite difference schemes, control the accumulation of computational error, and calibrate the parameters of the methods.
Second, the analytical representation of solutions in the form of hypergeometric functions makes it possible to efficiently compute initial conditions for numerical simulations in complex geometries, as well as to use asymptotic approximations that accelerate iterative processes in solvers. Thus, the results of this work establish a link between analytical solutions and applied computational methods.

2.3 Construction of solutions of Burgers hierarchy equations

To demonstrate the solution of some equations of the Burgers hierarchy, corresponding graphs are constructed. Several Cauchy problems for various k and n are considered.

Inverse Cole-Hopf transformations and dilations are performed. From expression (9) the function $F(z)$ is obtained. Then from (6) and (7) the expression for $u(x,t)$ is obtained.

The Cauchy problem (1.1) is a differential equation and initial conditions:

$$y^{(5)}(z) - 2zy'(z) + 2 \cdot 10 \cdot y(z) = 0,$$
$$y(0) = 18144, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0, \quad y^{(4)}(0) = 0.$$

Back substitution into the Cole-Hopf transform leads to Figure 1.

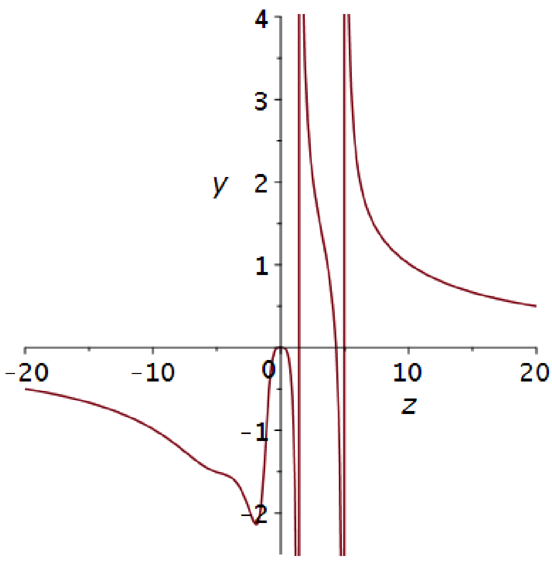

Figure 1: Function $F(z)$ for $k = 4$, $n = 10$

Inverse transformation to the strain transformations. A two-dimensional graph is plotted at a fixed time value $t = 1$ (Figure 2).

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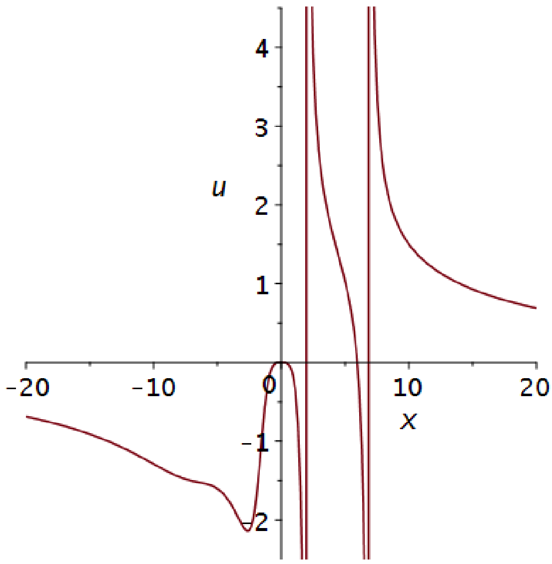

Figure 2: Function $u(x,t)$ for $k=4$, $n=10$, $t=1$

The Cauchy problem (1.2) is a differential equation and initial conditions:

$$y^{(6)}(x) - 2xy'(x) + 2\cdot 11\cdot y(x) = 0,$$
$$y(0)=0,\quad y'(0)=0,\quad y''(0)=0,\quad y'''(0)=0,\quad y^{(4)}(0)=0,\quad y^{(5)}(0)=-27720\cdot 128.$$

Back substitution into the Cole-Hopf transform leads to Figure 3.

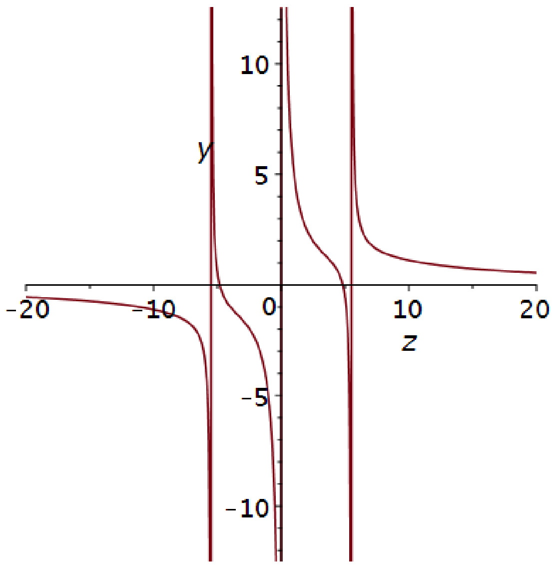

Figure 3: Function $F(z)$ for $k=5$, $n=11$

Inverse transformation to the strain transformations. A two-dimensional graph is plotted at a fixed time value $t=1$ (Figure 4).

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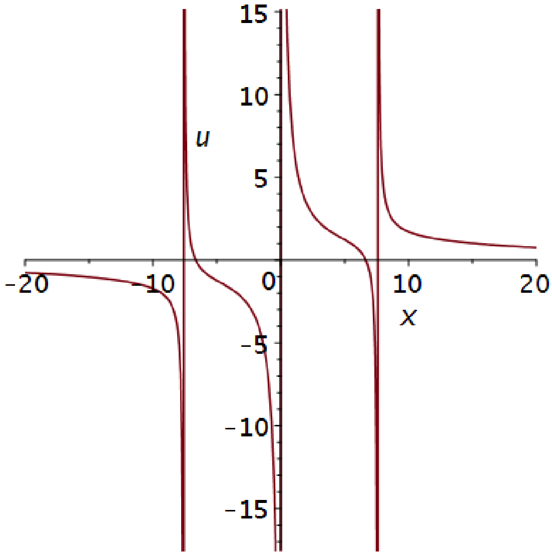

Figure 4: Function $u(x, t)$ for $k = 5$, $n = 11$, $t = 1$

For a fixed value of t, the functions $u(x, t)$ tend to infinity at the denominator's zeros, the number of which is equal to the degree of the ODE solution. This means that these functions have discontinuity points. If x tends to infinity, then the function $u(x, t)$ tends to 0. Also, for odd values of order k, the solutions will be symmetric about the origin. For even k, this dependence is not observed.

3 Conclusion

The paper presents some analytical solutions to ODEs obtained from the Burgers hierarchy. Initial conditions are used to obtain the generalized Hermite polynomials.

A lemma has been proven for the general hypergeometric equation using mathematical induction. It has shown that, using equivalent transformations, ODEs from the Burgers hierarchy can be reduced to a hypergeometric equation of type ${}_1F_q$. The solutions to this equation are expressed in terms of hypergeometric functions. The linear independence of the solutions to the equation for arbitrary k has been proven. A general solution formula has been obtained.

The initial conditions for the Cauchy problem have been found. A general formula has been written down for nonzero coefficients, for which the solutions coincide for any natural n and order k.

These exact solutions provide reliable benchmarks for verifying numerical schemes and quantifying the rate of convergence in computational models of nonlinear transport, and also help determine the initial conditions of Cauchy problems for equations whose solutions are found by numerical methods.

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References

[1] Kudryashov N.A. Special polynomials associated with the Burgers hierarchy. *Applied Mathematics and Computation*, vol. 218, no. 15, pp. 7972-7976, 2012.

[2] Davydov A.S. Kvantovaya mekhanika [Quantum Mechanics] *Ripol Klassik*, 1968 (in Russian).

[3] Geydarov T.G. O svyazi mezhdu nachalnymi momentami n-go poryadka normalnogo raspredeleniya sluchaynykh velichin i uravneniyami teploprovodnosti i nekotorye rekurrentnye formuly [On the connection between the initial moments of the n-th order of the normal distribution of random variables and the heat equations and some recurrence formulas] *Fundamentalnye i prikladnye issledovaniya: problemy i rezultaty*, no. 12, pp. 76-81, 2014 (in Russian).

[4] Kudryashov N.A. Generalized Hermite polynomials for the Burgers hierarchy and point vortices. *Chaos, Solitons & Fractals*, no. 151, pp. 111256, 2021.

[5] Burgers J.M. A mathematical model illustrating the theory of turbulence. *Advances in applied mechanics* vol. 1, pp. 171-199, 1948.

[6] Tasso H. Cole's ansatz and extension of Burgers equation, Report IPP6/142 Ber. *MPI fur Plasmaphysik (Garching)*, 1976.

[7] Sharma A.S., Tasso H. Connection between wave envelope and explicit solution of a nonlinear dispersive wave equation. *Max-Planck-Institut fuer Plasmaphysik*, 1977.

[8] Olver P.J. Evolution equations possessing infinitely many symmetries. *Journal of Mathematical Physics*, vol. 18, no. 6, pp. 1212-1215, 1977.

[9] Kudryashov N.A., Sinelshchikov D.I. Exact solutions of equations for the Burgers hierarchy. *Applied mathematics and computation*, vol. 215, no. 3, pp. 1293-1300, 2009.

[10] Cheb-Terrab E., Non-liouvillian solutions for second order Linear ODEs. *Proceedings of the 2004 international symposium on Symbolic and algebraic computation*, pp. 80-86, 2004.

[11] Graev M.I., Gelfand I.M., Retakh V.S. Obshchie gipergeometricheskie sistemy uravneniy i ryady gipergeometricheskogo tipa [General hypergeometric systems of equations and hypergeometric series] *Uspekhi matematicheskikh nauk*, vol. 47, no. 4(286), pp. 3-82, 1992 (in Russian).

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