*European Society of Computational Methods in Sciences and Engineering (ESCMSE)*

*Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM)*
vol. 19, no. 3-4, 2026, pp. 47-57
ISSN 1790–8140

# Conservation laws and exact solutions of the generalized Hasegawa–Kodama equation

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Received 8 August, 2025; accepted in revised form 14 August 2026

*Abstract:* The generalized Hasegawa-Kodama equation is considered. This equation is used to describe the propagation of pulses in nonlinear optical media. The studied equation generalizes the classical nonlinear Schrödinger equation. The generalized Hasegawa-Kodama equation accounts for complex nonlinear effects in the description of pulse propagation in nonlinear optical media. Conservation laws and corresponding first integrals for the equation under study have been constructed. Some exact solutions of the generalized Hasegawa-Kodama equation have been obtained. The derived solutions exhibit periodic behavior or take the form of optical solitons. Conserved quantities for solitary waves have been computed. The computed conserved quantities are used to verify the results of numerical simulations of optical pulses. The results of this work have practical significance for evaluating pulse parameters in nonlinear media.



*Keywords:* Hasegawa-Kodama equation; Optical solitons; Conservation laws; Nonlinear optics; Conserved quantities

*Mathematics Subject Classification:* 35Q55

## 1 Introduction

The classical Hasegawa-Kodama equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2\, q + i\,\varepsilon\,\left(q_{xxx} + 6\,|q|^2 q_x + 3\,(|q|^2)_x q\right) = 0 \tag{1}$$

was first introduced in [1], from Maxwell's equations and describes propagation of optical soliton in nonlinear media. This equation is integrable, and its initial value problem can be solved by inverse scattering transform.

In this work we investigate the generalized Hasegawa-Kodama equation in the following form

$$iq_t + aq_{xx} + b|q|^{2n}q + i(\alpha q_{xxx} + \beta|q|^{2n}q_x + \delta(|q|^{2n})_x q) = 0, \tag{2}$$

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where $q(x,t)$ is a complex-valued function, $x$ and $t$ are spatial and temporal coordinates, $a$, $b$, $\alpha$, $\beta$ and $\delta$ are real parameters of Eq. (2), $i^2 = -1$, $n \in \mathbb{N}$.

Eq. (2) is also called the generalized Sasa–Satsuma equation [2, 3, 4, 5, 6]. The choice of an arbitrary exponent $n$ in Eq. (2) allows to describe more complex nonlinear media [7, 8, 9, 10]. Although the generalized Hasegawa–Kodama equation is nonintegrable, it nevertheless admits some exact solutions.

Periodic solutions have been constructed for the generalized Hasegawa-Kodama equation at certain values of the parameter $n$, as well as bright optical solitons for arbitrary $n$ [4]. In paper [11] some exact solutions of the generalized Sasa-Satsuma equation has presented, which are expressed in terms of the Gaussian hypergeometric function. Using the trial equation method, optical solitons of Eq. (2) were constructed for $n = 1$ [12]. The study [13] investigated the fractional Sasa-Satsuma equation and obtained optical solitons for this model. A number of papers have been devoted to find solutions of the Hasegawa-Kodama equation [14, 15, 16, 17, 18].

Eq. (2) has m conservation laws if the following relation holds

$$\frac{\partial T_j}{\partial t} + \frac{\partial X_j}{\partial x} = 0, \qquad (j = 1, 2, .., m), \tag{3}$$

where $T \equiv T_j(q, q_t, q_x, \ldots, x, t)$ is the density and $X_j \equiv X(q, q_t, q_x, \ldots, x, t)$ is the flux.

Conservation laws serve as fundamental invariants in the study of partial differential equations [8, 19, 20, 21]. There are various methods for finding conservation laws of nonlinear partial differential equations [7]. In works [22, 23], a method for constructing conservation laws based on finding multipliers is used, such that when the equation is multiplied by these multipliers, it takes the form of a conservation law. For constructing conservation laws of generalized nonlinear Schrödinger equations, a modified multiplier method was proposed [24, 25, 26, 27, 28]. The essence of the method lies in representing the original complex equation as a system of two equations, finding multipliers by which the equations of the system are multiplied, and subsequently directly transforming the equations into the form of conservation laws. Using this method, we systematically derive these conservation laws for the generalized Hasegawa-Kodama equation.

Knowledge of conservation laws allows one to compute conserved quantities for optical solitons of the equations under study. These conserved quantities can be used to verify the results of numerical and neural network modeling of optical pulses.

The aim of this work is to find conservation laws and to determine exact solutions of Eq. (2) for arbitrary $n$.

The paper is structured as follows. Section 2 presents the derivation of conservation laws using the multiplier method. In Section 3, first integrals of the traveling wave equation are derived from the conservation laws. In Section 4 we construct exact solutions to the Hasegawa-Kodama equation for both $n = 1$ and arbitrary $n$. In Section 5 we calculate the conserved quantities for the optical soliton. Section 6 offers concluding remarks.

## 2 Conservation laws of Eq.(2)

In this work, the conservation laws were constructed using the direct computations method [29, 30, 31]. The essence of this method lies in selecting special-form multipliers for both the original equation and its adjoint counterpart, such that their combined expression can be represented in the form of conservation law.

To derive the conservation laws, we begin by writing both the Hasegawa-Kodama equation

$$i\, q_t + a\, q_{xx} + b\, q\, |q|^{2n} + i\left(\alpha\, q_{xxx} + \beta\, |q|^{2n}\, q_x + \delta\, (|q|^{2n})_x\, q\right) = 0 \tag{4}$$

and its complex-conjugate counterpart

$$-i\, q_t^* + a\, q_{xx}^* + b\, q^*\, |q|^{2n} - i\left(\alpha\, q_{xxx}^* + \beta\, |q|^{2n}\, q_x^* + \delta\, (|q|^{2n})_x\, q^*\right) = 0. \tag{5}$$

To derive the first conservation law, we multiply Eq. (4) by $q^*$ and Eq. (5) by $-q$, then add them together. As a result, we obtain

$$\begin{gathered} i\,(q_t\, q^* + q_t^*\, q) + a\,(q_{xx}\, q^* - q_{xx}^*\, q) + i\,\alpha\,(q_{xxx}\, q^* + q_{xxx}^*\, q) + \\ i\,\beta\, |q|^{2n}\,(q_x\, q^* + q_x^*\, q) + 2\, i\,\delta\,\left(|q|^{2n}\right)_x |q|^2 = 0. \end{gathered} \tag{6}$$

We divide Eq. (6) by $i$ and express it in conservation law form

$$\frac{\partial T_1}{\partial t} + \frac{\partial X_1}{\partial x} = 0, \tag{7}$$

where $T_1$ and $X_1$ take the form

$$\begin{gathered} T_1 = |q|^2, \\ X_1 = ia(q\, q_x^* - q^* q_x) + \alpha(q_{xx} q^* + q_{xx}^* q - |q_x|^2) + \frac{\beta + 2\delta n}{n+1} |q|^{2n+2}. \end{gathered} \tag{8}$$

We derive the second conservation law using the multiplier method. To achieve this, we multiply Eq. (4) by $q_x^*$ and Eq. (5) by $q_x$, then we sum up the obtained expressions. The outcome takes the form

$$\begin{gathered} i(q_t q_x^* - q_t^* q_x) + a(q_{xx} q_x^* + q_{xx}^* q_x) + b|q|^{2n}(q q_x^* + q^* q_x) + \\ i\alpha(q_{xxx} q_x^* - q_{xxx}^* q_x) + i\delta(|q|^{2n})_x(q q_x^* - q^* q_x) = 0. \end{gathered} \tag{9}$$

Eq. (9) can be reduced to a conservation law form under constraint $\delta = 0$. In this case the second conservation law takes the form

$$\frac{\partial T_2}{\partial t} + \frac{\partial X_2}{\partial x} = 0, \tag{10}$$

where $T_2$ and $X_2$ are determined by the formulas

$$\begin{gathered} T_2 = \frac{i}{2}(q_x^* q - q^* q_x), \\ X_2 = \frac{i}{2}(q^* q_t - q_t^* q) + a|q_x|^2 + \frac{b}{n+1}|q|^{2n+2} + i\alpha(q_{xx} q_x^* - q_{xx}^* q_x). \end{gathered} \tag{11}$$

Now we construct the third conservation law. For this purpose, we multiply Eq. (4) by $q_t^*$ and Eq. (5) by $q_t$. Adding these expressions yields the following equation

$$\begin{gathered} a(q_{xx} q_t^* + q_{xx}^* q_t) + b|q|^{2n}(q q_t^* + q^* q_t) + i\alpha(q_{xxx} q_t^* - q_{xxx}^* q_t) + \\ i\beta|q|^{2n}(q_x q_t^* - q_x^* q_t) + i\delta(|q|^{2n})_x(q q_t^* - q^* q_t) = 0. \end{gathered} \tag{12}$$

Under constraint $\delta = 0$, Eq. (12) can be expressed as the conservation law

$$\frac{\partial T_3}{\partial t} + \frac{\partial X_3}{\partial x} = 0, \tag{13}$$

where $T_3$ and $X_3$ are the following

$$
\begin{aligned}
T_3 = -a|q_x|^2 + \frac{b}{n+1}|q|^{2n+2} + \frac{i\alpha}{2}(q_{xx}^* q_x - q_{xx} q_x^*) + \\
\frac{i\beta}{2(n+1)}|q|^{2n}(q^* q_x - q_x^* q), \\
X_3 = a(q_x q_t^* + q_x^* q_t) + i\alpha(q_{xx} q_t^* - q_{xx}^* q_t) + \frac{i\alpha}{2}(q_{xt} q_x^* - q_{xt}^* q_x) + \\
\frac{i\beta}{2(n+1)}|q|^{2n}(q_t^* q - q_t q^*).
\end{aligned} \tag{14}
$$

The conservation laws derived in this section will subsequently be used to determine first integrals of the system and compute conserved quantities.

## 3 First integrals corresponding to Eq.(2)

In this section, we derive first integrals for the system of ordinary differential equations obtained from Eq. (2) by the reduction to traveling wave variables

$$
q(x,t) = y(z)\,\mathrm{e}^{i\,(k\,x-\omega\,t)}, \quad z = x - C_0\,t. \tag{15}
$$

For this purpose, we use a method based on use of conservation laws [31]. To do this, in the first conservation law (7), we reduce to traveling wave variables, and then the resulting expression is integrated with respect to the variable $z$. A similar approach is applied to the second conservation law (10). The third conservation law (13), however, will not yield new first integrals after reduction to traveling wave variables, it produce an expression already known from the second conservation law, up to multiplication by a constant.

Let us substitute (15) into the conservation law (7) and integrate the resulting equation with respect to $z$. We obtain the first integral

$$
2aky^2 + \alpha(2yy_{zz} - y_z^2 - 3k^2y^2) + \frac{\beta + 2\delta n}{n+1}y^{2n+2} - C_0y^2 = C_1, \tag{16}
$$

where $C_1$ is an arbitrary constant.

From the second conservation law (10) with the restriction on parameter $\delta = 0$, the first integral of the following form is obtained

$$
a(y_z^2 + k^2y^2) + \frac{b}{n+1}y^{2n+2} - 2\alpha(2ky_z^2 + k^3y^2 - kyy_{zz}) + (\omega - C_0k)y^2 = C_2, \tag{17}
$$

where $C_2$ is an arbitrary constant.

Let us introduce the substitution

$$
y(z) = f^2(z) \tag{18}
$$

in Eqs. (16) and (17). In Eq. (16), it is also necessary to set $\delta = 0$. Then, the system of Eqs. (16) and (17) takes the form

$$
\begin{cases}
4\alpha f^3 f_{zz} + (2ak - 3\alpha k^2 - C_0)f^4 + \frac{\beta}{n+1}f^{4(n+1)} = C_1, \\
4\alpha k f^3 f_{zz} + (4a - 12\alpha k)f^2 f_z^2 + (ak^2 - 2\alpha k^3 + \omega - C_0k)f^4 + \frac{b}{n+1}f^{4(n+1)} = C_2.
\end{cases} \tag{19}
$$

The system of Eqs. (19) will be consistent under the following conditions

$$\begin{cases} a = 3\,\alpha\,k, \\ \omega = 2\,\alpha\,k^3, \\ b = \beta\,k, \\ C_2 = C_1 k. \end{cases} \tag{20}$$

Now, instead of system (19), we can consider a single equation, which we write in the form

$$4\alpha f^3 f_{zz} + (3\alpha k^2 - C_0) f^4 + \frac{\beta}{n+1} f^{4(n+1)} = C_1. \tag{21}$$

Let us multiply Eq. (21) by $\dfrac{f_z}{f^3}$ and integrate with respect to the variable $z$. The resulting expression takes the form

$$2\alpha f_z^2 + \frac{3\alpha k^2 - C_0}{2} f^2 + \frac{\beta}{2(n+1)(2n+1)} f^{4n+2} + \frac{C_1}{2f^2} + \frac{C_3}{2} = 0, \tag{22}$$

where $C_3$ is an arbitrary constant.

We multiply the resulting Eq. (22) by $2f^2$ and revert to the original variable $y$. The equation then takes the form

$$\alpha y_z^2 + (3\alpha k^2 - C_0) y^2 + \frac{\beta}{(n+1)(2n+1)} y^{2n+2} + C_1 + C_3 y = 0. \tag{23}$$

Eq. (23) is not integrable in the general case. However, it admits certain exact solutions under specific constraints on the equation parameters.

## 4 Exact solutions of Eq. (2)

Let's construct some exact solutions to Eq. (23). Eq. (23) admits an optical soliton solution for arbitrary values of parameter $n$ when the integration constants satisfy $C_1 = C_3 = 0$. Under these conditions, Eq. (23) reduces to the following form

$$\alpha y_z^2 + (3\alpha k^2 - C_0) y^2 + \frac{\beta}{(n+1)(2n+1)} y^{2n+2} = 0. \tag{24}$$

In Eq. (24), we perform a change of variables of the form

$$y(z) = V^{\frac{1}{n}}(z) \tag{25}$$

and rewrite it as

$$V_z^2 - M V^2 + N V^4 = 0, \tag{26}$$

where

$$M = \frac{(C_0 - 3\alpha k^2) n^2}{\alpha}, \quad N = \frac{\beta n^2}{\alpha(n+1)(2n+1)}. \tag{27}$$

The solution of Eq. (26) has the form of a bright optical soliton and can be represented as

$$V(z) = \frac{4\,M\,\mathrm{e}^{\sqrt{M}\,(z-z_0)}}{4\,M\,N + \mathrm{e}^{2\,\sqrt{M}(z-z_0)}}, \tag{28}$$

where $z_0$ is an arbitrary constant.

Optical solitons remain nonsingular and physically bounded in the parameter domain

$$\begin{cases} \alpha \cdot (C_0 - 3\alpha k^2) > 0, \\ \alpha \cdot \beta > 0. \end{cases} \tag{29}$$

The corresponding solution of Eq. (24) in this case takes the form

$$y_1(z) = \left( \frac{4\,M\,\mathrm{e}^{\sqrt{M}\,(z-z_0)}}{4\,M\,N + \mathrm{e}^{2\,\sqrt{M}(z-z_0)}} \right)^{\frac{1}{n}}. \tag{30}$$

Solution (30) (after the variable change $z = x - C_0 t$) is illustrated in Fig. 1 at $n = 4, z_0 = 5.0, C_0 = 6.0, M = 4.0$ and $N = 3.0$.

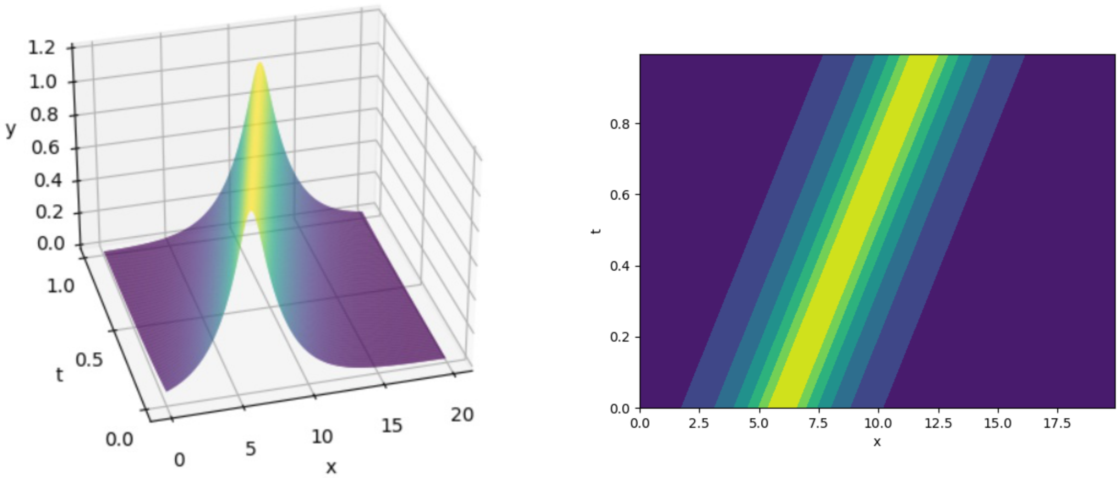


Figure 1: Solution (30) at $n = 4, z_0 = 5.0, C_0 = 6.0, M = 4.0$ and $N = 3.0$

The optical soliton solution of Eq. (2) for arbitrary $n$ can be represented in the form

$$q_1(x,t) = \left( \frac{4\,M\,\mathrm{e}^{\sqrt{M}\,(x-C_0 t-z_0)}}{4\,M\,N + \mathrm{e}^{2\,\sqrt{M}(x-C_0 t-z_0)}} \right)^{\frac{1}{n}} \mathrm{e}^{i\,(k\,x-2\alpha k^3 t)}. \tag{31}$$

Let us consider Eq. (23) for case $n = 1$. In this case, it takes the form

$$\alpha y_z^2 - (C_0 - 3\alpha k^2) y^2 + \frac{\beta}{6} y^4 + C_1 + C_3 y = 0. \tag{32}$$

The general solution of Eq. (32) can be written in terms of the Jacobi elliptic function as follows

$$y_2(z) = \frac{Y_1\,(Y_4 - Y_3)\,\mathrm{sn}^2\,\{S_1\,(z-z_0);\,k_1\} + Y_4\,(Y_3 - Y_1)}{(Y_4 - Y_3)\,\mathrm{sn}^2\,\{S_1\,(z-z_0);\,k_1\} + Y_3 - Y_1}, \tag{33}$$

where $z_0$ is an arbitrary constant and $Y_1, Y_2, Y_3$ and $Y_4$ are the roots of the equation

$$\frac{\beta}{6}y^4 - (C_0 - 3\alpha k^2)y^2 + C_3 y + C_1 = \\ \frac{\beta}{6}(y - Y_1)(y - Y_2)(y - Y_3)(y - Y_4) = 0. \tag{34}$$

The parameters $S_1$ and $k_1$ are given by

$$S_1 = \sqrt{\frac{\beta}{24\alpha}(Y_4 - Y_2)(Y_3 - Y_1)}, \\ k_1 = \sqrt{\frac{(Y_2 - Y_1)(Y_4 - Y_3)}{(Y_4 - Y_2)(Y_3 - Y_1)}}. \tag{35}$$

Solution (33)(after the variable change $z = x - C_0 t$) is illustrated in Fig. 2 at $z_0 = 2.0, C_0 = 2.0, \alpha = 1.0, \beta = 54.0, Y_1 = 1.0, Y_2 = 1.2, Y_3 = 2.0$ and $Y_4 = 2.2$.

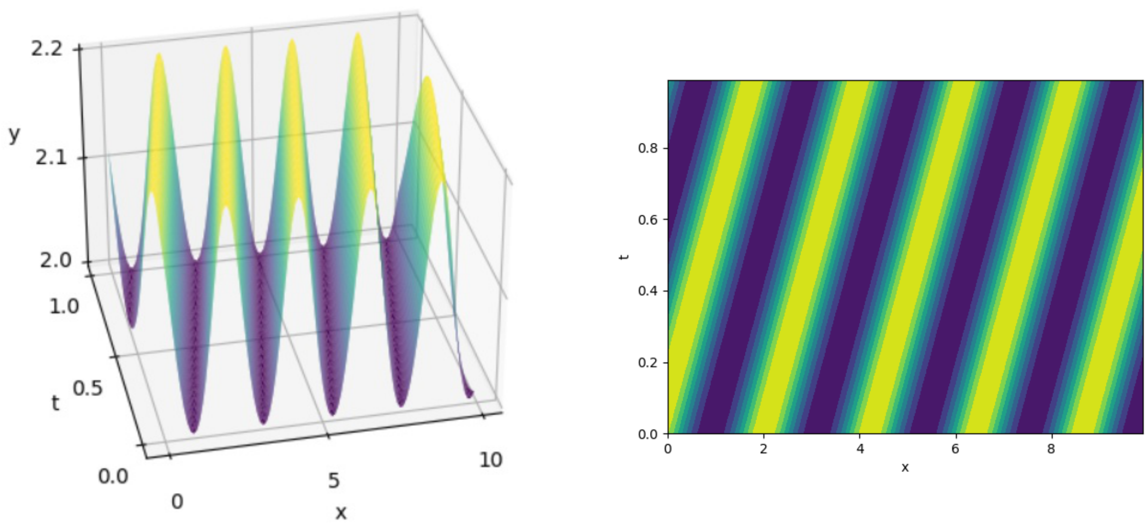


Figure 2: Solution (33) at $z_0 = 2.0, C_0 = 2.0, \alpha = 1.0, \beta = 54.0, Y_1 = 1.0, Y_2 = 1.2, Y_3 = 2.0$ and $Y_4 = 2.2$

The corresponding periodic solution of Eq. (2) can be expressed as

$$q_2(x,t) = \left(\frac{Y_1(Y_4 - Y_3)\,\mathrm{sn}^2\{S_1(x - C_0 t - z_0);\,k_1\} + Y_4(Y_3 - Y_1)}{(Y_4 - Y_3)\,\mathrm{sn}^2\{S_1(x - C_0 t - z_0);\,k_1\} + Y_3 - Y_1}\right) \mathrm{e}^{i\,(k\,x - 2\alpha k^3 t)}. \tag{36}$$

In the case of $C_1 = C_3 = 0$, Solution (36) transforms into a solitary wave of the form.

Eq. (23) also has solutions for noninteger values of $n$. Specifically, for $n = \frac{1}{2}$, it admits a periodic solution expressed in terms of elliptic functions [4].

## 5 Conserved quantities for solitary wave solution (31) of Eq. (2)

Knowledge of conservation laws enables us to find conserved quantities for optical solitons. To derive them, we integrate the conservation law

$$\frac{\partial T_j}{\partial t} + \frac{\partial X_j}{\partial x} = 0 \tag{37}$$

over $x$ from $-\infty$ to $+\infty$ and obtain the following invariants

$$I_j = \int_{-\infty}^{+\infty} T_j\, dx = constant, \tag{38}$$

which is independent of $t$ for any soliton solution $q(x,t)$.

In this section, we compute the conserved quantities for the exact solution (31). To do this, we first evaluate an auxiliary integral that will be used in the derivations

$$\begin{gathered}\Omega_k = \int_{-\infty}^{+\infty} |q_1|^{2k} dx = \int_{-\infty}^{+\infty} y_1^{2k} dz = \int_{-\infty}^{+\infty} \left( \frac{4\,M\,e^{\sqrt{M}\,(z-z_0)}}{4\,M\,N + e^{2\,\sqrt{M}\,(z-z_0)}} \right)^{\frac{2k}{n}} dz = \\ \left| \frac{2k}{n} = p \right| = \frac{(\sqrt{4\,M\,N})^p}{N^p \sqrt{M}} \int_0^{+\infty} \frac{\varphi^{p-1}}{(1+\varphi^2)^p} d\varphi = \frac{(\sqrt{M})^{p-1}}{(\sqrt{N})^p} \cdot \frac{\sqrt{\pi}\,\Gamma(\frac{p}{2})}{\Gamma(\frac{p+1}{2})}.\end{gathered} \tag{39}$$

Let us compute the conserved quantity corresponding to the first conservation law (7). The result takes the form

$$I_1 = \int_{-\infty}^{+\infty} T_1\, dx = \int_{-\infty}^{+\infty} |q|^2\, dx = \Omega_1. \tag{40}$$

The obtained quantity (40) has a clear physical meaning. It represents the power of the optical soliton (31).

Let us derive the conserved quantity corresponding to the second conservation law (10). For the optical soliton (31), this quantity takes the form

$$I_2 = \int_{-\infty}^{+\infty} T_2\, dx = \int_{-\infty}^{+\infty} \frac{i}{2}(q_x^* q - q^* q_x)\, dx = k\,\Omega_1. \tag{41}$$

The calculated quantity represents the soliton momentum.

Now let us derive the third conserved quantity from equation (13), taking into account the constraints (20). We express it in the form

$$\begin{gathered}I_3 = \int_{-\infty}^{+\infty} T_3\, dx = \int_{-\infty}^{+\infty} \left( -3\,\alpha\, k |q_x|^2 + \frac{\beta k}{n+1}|q|^{2n+2} + \frac{i\alpha}{2}(q_{xx}^* q_x - q_{xx} q_x^*) + \right. \\ \left. \frac{i\beta}{2(n+1)}|q|^{2n}(q^* q_x - q_x^* q) \right) dx = 2k(2\alpha k^2 - C_0)\,\Omega_1 + \frac{\beta k(n+2)}{(n+1)(2n+1)}\,\Omega_{n+1}.\end{gathered} \tag{42}$$

The physical meaning of this conserved quantity is the energy of the optical soliton (31).

The obtained conserved quantities are useful for evaluating these characteristics measured during pulse propagation in optical fibers.

## 6 Conclusions

In this paper, we have considered the generalized Hasegawa-Kodama equation for an arbitrary value of the exponent $n$. Three conservation laws were constructed for the equation using the multiplier method. The first conservation law was found without any restrictions on the equations parameters, while the second and third were obtained under certain constraints. First integrals of the system of ordinary differential equations derived from the Hasegawa-Kodama equation were constructed using the first two conservation laws. Exact solutions of the equation were then obtained. For an arbitrary exponent $n$, a solution in the form of a bright optical soliton was constructed. Additionally, a periodic solution was found for the parameter value $n = 1$. Subsequently, the conserved quantities for the optical soliton were computed.

The calculated conserved quantities of optical solitons can be used to estimate physical quantities measured during long-distance information transmission. Furthermore, conserved quantities are used to verify the results of numerical simulations of optical pulses. They can help identify computational errors and select the most accurate scheme for solving the problem. Conserved quantities are also used to compare the results of neural network modeling.

## Acknowledgment

This research was supported by the Ministry of Science and Higher Education of the Russian Federation according to the state task project No. FSWU-2026-0006.

The author wishes to thank the anonymous referees for their careful reading of the manuscript and their fruitful comments and suggestions.

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