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On recent developments in the theory of dynamic Hardy-type inequalities on time scales

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Abstract: The main objective of this work is to give several dynamic inequalities of Hardy-type on time scales which represent the unified approach of the classical integral and discrete inequalities. These results include a new reformulation of Hardy-type inequality which is valid for $p = 1$ and Hardy-type inequalities with a negative parameter $p < 0$. Furthermore, we present an inequality involving the Hardy-Steklov operator which is a generalization of Hardy operator, where the lower and upper limits of the integration are increasing functions. In addition, we show the characterizations of weighted functions that are suitable for the validity of the weighted Hardy-type inequality with two different spaces ℓ_p, ℓ_q for $1 < p \leq q < \infty$ and two different weights involving a nonnegative general kernel. Finally, the last aim is to get a global measure of the dispersion of the function f around its integral mean $(1/x)\int_a^b f(x)\Delta x$ which gives the standard deviation when $p = 2$.

Keywords: Hardy type inequalities, Time scale, Hölder's inequality, Jensen's inequality, Hardy-Steklov operator, The global measure of the dispersion.

Mathematics Subject Classification: 26D10, 26D15, 34N05, 47B38, 39A12

1 Introduction

In 1920, Hardy [11] proved the discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^{p} \leq \left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a^{p}(n), \quad p>1, \tag{1.1}$$

where $a(n) \geq 0$ for $n \geq 1$ and $a(n) \in l^p(\mathbb{N})$ (i.e. $\sum_{n=1}^{\infty} a^p(n) < \infty$). This inequality has been discovered in his attempt to give an elementary proof of Hilbert's inequality for double series that was known at that time. In [13, Theorem A] Hardy proved the integral version of (1.1), by using the calculus of variations, which states that for $f \geq 0$ and integrable over any finite interval $(0, x)$, where $x \in (0, \infty)$ and $f \in L^p(0, \infty)$ and $p > 1$, then

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^{x}f(t)\mathrm{d}t\right)^{p}\mathrm{d}x \leq \left(\frac{p}{p-1}\right)^{p}\int_0^{\infty}f^{p}(x)\mathrm{d}x. \tag{1.2}$$

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In both the continuous and discrete cases, the constant $C = (p/(p-1))^p$ is sharp and cannot be replaced by a smaller value without losing its validity for certain sequences and functions.

In 1925, Hardy [13, Theorem B] generalized (1.1) by replacing the standard arithmetic mean $\sum_{i=1}^{n} a(i)/n$ with a more general arithmetic mean with general weights. In fact, he proved that if $p > 1$, $\lambda(n) > 0$, $a(n) \geq 0$ and $\Lambda(n) = \sum_{i=1}^{n} \lambda(i)$ for $n \geq 1$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^p(n)} \left(\sum_{i=1}^{n} \lambda(i) a(i) \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) a^p(n). \tag{1.3}$$

Copson [9] generalized (1.3) and proved that if $p > 1$, $a(i) \geq 0$, $\lambda(i) > 0$ $\forall i \in \mathbb{N}$ and $c > 1$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=1}^{n} \lambda(i) a(i) \right)^p \leq \left(\frac{p}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n). \tag{1.4}$$

Also, he proved that if $p > 1$ and $0 \leq c < 1$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=n}^{\infty} \lambda(i) a(i) \right)^p \leq \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n). \tag{1.5}$$

Leindler [14] proved that if $\sum_{i=n}^{\infty} \lambda(i) < \infty$, $p > 1$ and $0 \leq c < 1$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{(\Lambda^*(n))^c} \left(\sum_{i=1}^{n} \lambda(i) a(i) \right)^p \leq \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda(n) (\Lambda^*(n))^{p-c} a^p(n),$$

where $\Lambda^*(n) = \sum_{i=n}^{\infty} \lambda(i)$, and if $1 < c \leq p$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{(\Lambda^*(n))^c} \left(\sum_{i=n}^{\infty} \lambda(i) a(i) \right)^p \leq \left(\frac{p}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) (\Lambda^*(n))^{p-c} a^p(n).$$

Bennett [4] and Leindler [14] proved the reversed forms of the inequalities (1.4) and (1.5). In particular they proved that if $c \leq 0 < p < 1$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=n}^{\infty} \lambda(i) a(i) \right)^p \geq \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n),$$

and if $c > 1 > p > 0$ and if $\Lambda_n \to \infty$ as $n \to \infty$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda_n^c} \left(\sum_{i=1}^{n} \lambda(i) a(i) \right)^p \geq \left(\frac{pL}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda_n^{p-c} a^p(n),$$

where $L = \inf \frac{\lambda(n)}{\lambda(n+1)}$.

Copson [10, Theorems 1 and 3] showed that the integral counterparts of the inequalities (1.4) and (1.5) also hold. In particular, he proved that if $p \geq 1$, $c > 1$ and $\lambda(s) > 0$, $g(s) \geq 0$ on $(0, \infty)$ with $\Lambda(t) = \int_0^t \lambda(s) \mathrm{d}s$ and $\Phi(t) = \int_0^t \lambda(s) g(s) \mathrm{d}s$, then

$$\int_0^{\infty} \frac{\lambda(t)}{\Lambda^c(t)} \Phi^p(t) \mathrm{d}t \leq \left(\frac{p}{c-1} \right)^p \int_0^{\infty} \frac{\lambda(t)}{\Lambda^{c-p}(t)} g^p(t) \mathrm{d}t,$$

and if $p>1$, $0\le c<1$, then

$$\int_0^\infty \frac{\lambda(t)}{\Lambda^c(t)}\Psi^p(t)\mathrm{d}t \le \left(\frac{p}{1-c}\right)^p \int_0^\infty \frac{\lambda(t)}{\Lambda^{c-p}(t)} g^p(t)\mathrm{d}t,$$

where $\Psi(t)=\int_t^\infty \lambda(s)g(s)\mathrm{d}s$.

The first weight version of the classical Hardy inequality was proved by Hardy himself [12, Theorem B] and given by

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p x^\gamma \mathrm{d}x \le \left(\frac{p}{p-1-\gamma}\right)^p \int_0^\infty x^\gamma f^p(x)\mathrm{d}x, \quad \text{for} \quad p>1,\ \gamma<p-1, \tag{1.6}$$

and

$$\int_0^\infty \left(\frac{1}{x}\int_x^\infty f(t)\mathrm{d}t\right)^p x^\gamma \mathrm{d}x \le \left(\frac{p}{1-(p-\gamma)}\right)^p \int_0^\infty x^\gamma f^p(x)\mathrm{d}x, \quad \text{for} \quad p>1,\ p<\gamma+1,$$

where f is a measurable and nonnegative function on $(0,\infty)$.

In 1972, Muckenhoupt [15] generalized (1.6) and characterized the weights such that the inequality

$$\left(\int_0^\infty u^p(x)\left(\int_0^x f(t)\mathrm{d}t\right)^p \mathrm{d}x\right)^{1/p} \le C\left(\int_0^\infty v^p(x) f^p(x)\mathrm{d}x\right)^{1/p}, \tag{1.7}$$

holds for all measurable functions $f\ge 0$ and the constant $C>0$ is independent of f (here $1<p<\infty$). The characterization reduces to the condition that the nonnegative functions u and v satisfy

$$\sup_{x>0}\left(\int_x^\infty u^p(t)\mathrm{d}t\right)^{1/p}\left(\int_0^x v^{-p^*}(t)\mathrm{d}t\right)^{1/p^*} = K<\infty, \quad p^*=\frac{p}{p-1},$$

and $K\le C\le p^{1/p}(p^*)^{1/p^*}K$.

In 1978, Bradley [7] studied (1.7) in the different spaces $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ when $1\le p\le q\le\infty$, and gave new characterizations of weights such that the general inequality

$$\left(\int_0^\infty u^q(x)\left(\int_0^x f(t)\mathrm{d}t\right)^q \mathrm{d}x\right)^{1/q} \le C\left(\int_0^\infty v^p(x) f^p(x)\mathrm{d}x\right)^{1/p},$$

holds for all measurable functions $f\ge 0$ and the constant $C>0$ is independent of f (here $1\le p\le q\le\infty$). The characterization reduces to the condition that the nonnegative functions u and v satisfy

$$\sup_{x>0}\left(\int_x^\infty u^q(t)\mathrm{d}t\right)^{1/q}\left(\int_0^x v^{-p^*}(t)\mathrm{d}t\right)^{1/p^*} = A<\infty,$$

and $A\le C\le p^{1/q}(p^*)^{1/p^*}A$, for $1<p<q<\infty$ and $A=C$ if $p=1$ and $q=\infty$.

In 1990, Opic and Kufner [16] established the finite version of weighted Hardy-type inequality and proved that the inequality

$$\left(\int_a^b u(x)\left(\int_a^x f(t)\mathrm{d}t\right)^q \mathrm{d}x\right)^{1/q} \le C\left(\int_a^b v(x) f^p(x)\mathrm{d}x\right)^{1/p}, \tag{1.8}$$

holds for all nonnegative functions f and u, v are measurable positive functions in (a,b), $-\infty\le a<b<\infty$ and $1<p\le q<\infty$, if and only if the following condition holds

$$B=\sup_{a<x<b}\left(\int_x^b u(t)\mathrm{d}t\right)^{1/q}\left(\int_a^x v^{1-p^*}(t)\mathrm{d}t\right)^{1/p^*}<\infty.$$

Moreover, the estimate for the constant $C > 0$ in (1.8) is given by

$$B \leq C \leq \left(1 + \frac{q}{p^*}\right)^{1/q} \left(1 + \frac{p^*}{q}\right)^{1/p^*} B.$$

In 1991, Sinnamon [21] characterized the weights such that the inequality (1.8) is valid for $0 < q < 1 < p < \infty$. The characterization reduces to the condition that the nonnegative functions u and v satisfy

$$D = \left[\int_{\mathbf{a}}^{b} \left(\int_{x}^{b} u(t)\mathrm{d}t\right)^{r/p} \left(\int_{a}^{x} v^{1-p^*}(t)\mathrm{d}t\right)^{r/p^*} u(x)\mathrm{d}x\right]^{\frac{1}{r}} < \infty,$$

where $1/r = 1/q - 1/p$ and $(p^*)^{1/p^*} q^{1/p} D \leq C \leq (p^*)^{1/r} q^{1/p} D$.

Our general aim in this work is to unify the integral and discrete inequalities of Hardy-type in a single inequality with a general domain called a time scale $\mathbb{T}$ which is defined as a nonempty closed subset of the real numbers $\mathbb{R}$, to avoid repeating its proving twice (once for the continuous case and another for the discrete case). As special cases of our results on time scales, when $\mathbb{T} = \mathbb{R}$, we get the continuous analogue while $\mathbb{T} = \mathbb{N}$, the discrete cases can be obtained. Also, we can get the quantum analogue when $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$.

This work aims to discuss the diversity of recent developments in Hardy-type inequalities through the theory of time scales. We present new contributions in this area, aiming to highlight advances that unify discrete and continuous analysis. The ideas presented include: a new reformulation of Hardy-type inequality such that it holds for $p = 1$ and for some negative parameters $p < 0$. In addition, we discuss the characterizations of the weighted functions for validity of the weighted Hardy inequality with kernel in two different spaces ℓ_p, ℓ_q, $1 < p \leq q < \infty$ and the dynamic inequalities involving the general operators "Hardy-Steklov operator and Copson Steklov operator". Finally, we present the new meaning of Hardy-type inequality to get the global measure of the dispersion of the function around its mean.

The paper is organized as follows. Section 2 collects preliminary definitions, theorems and lemmas from time scale calculus that will be needed in proving the main results. Section 3 discusses a great idea which is represented as "how to derive the form of dynamic inequality on time scales". Section 4 contains our main results involving the recent developments of dynamic Hardy-type inequalities on time scales. Section 5 contains an application of a dynamic inequality of Hardy-type to show the oscillation properties of a second order dynamic equation. Finally, in Section 6, we present some achieved numerical examples on our results.

2 Preliminaries and Basic Lemmas

In this section, we briefly present some basic definitions and facts from time scales calculus we will use in the sequel. For more details, we refer the reader to [6].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology on $\mathbb{R}$. The forward jump operator and the backward jump operator σ, $\rho : \mathbb{T} \to \mathbb{T}$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The function $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) := \sigma(t) - t$ is known as the graininess function. For a point $t \in \mathbb{T}$, we say t is right-scattered (left-scattered) if $\sigma(t) > t$ ($\rho(t) < t$). Points that are right-scattered and left-scattered at the same time are called isolated. Similarly, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$ ($t > \inf \mathbb{T}$ and $\rho(t) = t$), then t is called right-dense (left-dense). Points that are right-dense and left-dense at the same time are called dense.

The time scale interval $[a,b]_{\mathbb{T}}$ is defined by $[a,b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$ with $a, b \in \mathbb{T}$; open intervals and half open intervals are defined similarly. For a function $f : \mathbb{T} \to \mathbb{R}$, we define $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ and $f^{\rho} : \mathbb{T} \to \mathbb{R}$ by $f^{\sigma}(t) = f(\sigma(t))$ and $f^{\rho}(t) = f(\rho(t))$, respectively, for all $t \in \mathbb{T}$. Moreover, we let

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}),\ \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition 1 (See [6, Definition 1.10]) *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we define $f^{\Delta}(t)$ to be the number, provided it exists, with the property that, for any $\varepsilon > 0$, there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ of t, for some $\delta > 0$, such that*

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in U \backslash \{\sigma(t)\}.$$

In this case, we say $f^{\Delta}(t)$ is the delta derivative of f at t.

Theorem 1 (See [6, Theorem 1.20]) *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$. Then*

1. *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*
$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$
2. *For any constant $\alpha, \alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*
$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$
3. *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t and "the product rule" is given by*
$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$
4. *If $g(t)g(\sigma(t)) \neq 0$, then the quotient $f/g : \mathbb{T} \to \mathbb{R}$ is differentiable at t and "the quotient rule" is defined by*
$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Theorem 2 (See [6, Theorem 1.76]) *If $f^{\Delta} \geq 0$ ($f^{\Delta} \leq 0$), then f is nondecreasing (nonincreasing).*

Theorem 3 (Chain Rule, see [6, Theorem 1.87]) *Assume $v : \mathbb{R} \to \mathbb{R}$ is continuous, $v : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$, and $u : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with*

$$(u \circ v)^{\Delta}(t) = u'(v(c))v^{\Delta}(t). \tag{2.1}$$

Definition 2 (See [6, Definition 1.58]) *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) for every left-dense points $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$.*

Definition 3 (See [6, Definition 1.71]) *A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided*

$$F^{\Delta}(t) = f(t) \quad \text{holds for all } t \in \mathbb{T}^{\kappa}.$$

In this case, the Cauchy integral of f is defined by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all } r, s \in \mathbb{T}.$$

Theorem 4 (See [6, Theorem 1.74]) *Every rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then*

$$F(t) = \int_{t_0}^{t} f(\tau)\Delta\tau$$

is an antiderivative of f.

Theorem 5 (See [6, Theorem 1.77]) *If $a, b, c \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $u, v \in \mathrm{C}_{\mathrm{rd}}([a,\, b]_{\mathbb{T}}, \mathbb{R})$, then*

1. $\int_a^b [\alpha u(t) + \beta v(t)]\, \Delta t = \alpha \int_a^b u(t)\Delta t + \beta \int_a^b v(t)\Delta t$,
2. $\int_a^b u(t)\Delta t = -\int_b^a u(t)\Delta t$,
3. $\int_a^b u(t)\Delta t = \int_a^c u(t)\Delta t + \int_c^b u(t)\Delta t$,
4. $\int_a^a u(t)\Delta t = 0$,
5. $\left|\int_a^b u(t)\Delta t\right| \le \int_a^b |u(t)|\, \Delta t$,
6. $u(t) \ge 0$ *for all* $t \in [a,\, b]_{\mathbb{T}}$*, implies* $\int_a^b u(t)\Delta t \ge 0$,
7. *the integration by parts rule*

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t \tag{2.2}$$

holds.

Lemma 1 (Hölder inequality, see [6, Theorem 6.13]) *If $a, b \in \mathbb{T}$ and $f, g \in \mathrm{C}_{\mathrm{rd}}\left([a,b]_{\mathbb{T}}, \mathbb{R}^{+}\right)$, then*

$$\int_a^b f(t)g(t)\Delta t \le \left[\int_a^b f^{\gamma}(t)\Delta t\right]^{\frac{1}{\gamma}} \left[\int_a^b g^{\nu}(t)\Delta t\right]^{\frac{1}{\nu}}, \tag{2.3}$$

where $\gamma > 1$ and $1/\gamma + 1/\nu = 1$. The Hölder inequality (2.3) *is reversed with different sign provided that either $p < 0$ or $0 < p < 1$.*

Lemma 2 (Jensen's inequality, see [6, Theorem 6.17]) *Let $a, b \in \mathbb{T}$ with $b > a$ and $c, d \in \mathbb{R}$. If $f \in C_{\mathrm{rd}}\left([a,b]_{\mathbb{T}}, (c,d)\right)$ and $F \in C\left((c,d), \mathbb{R}\right)$ is convex, then*

$$F\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \le \frac{\int_a^b F(f(t))\Delta t}{b-a}. \tag{2.4}$$

3 Main Idea for Deriving Inequalities on Time Scales

To state the main idea to get the form of the unified approach of the continuous and discrete inequalities, we have some steps:

First, by using the definition of the forward jump operator σ, backward jump operator ρ, and the graininess function μ as in Section 2, we can easily find that for the continuous case (when $\mathbb{T} = \mathbb{R}$), $\sigma(t) = \rho(t) = t$, and $\mu(t) = \sigma(t) - t = 0$ for any point $t \in \mathbb{R}$; and for the discrete case (when $\mathbb{T} = \mathbb{N}$), $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) = \sigma(t) - t = 1$ for any point $t \in \mathbb{N}$.

Second, the integration on an arbitrary time scale $\mathbb{T}$ is defined by

$$\int_a^b f(x)\Delta x.$$

This form can be transformed to the continuous analogue (when $\mathbb{T} = \mathbb{R}$) as follows:

$$\int_a^b f(x)\Delta x := \int_a^b f(x)\mathrm{d}x,$$

and the discrete form (when $\mathbb{T} = \mathbb{N}$) is given by

$$\int_a^b f(x)\Delta x := \sum_{x\in[a,b)} \mu(x)f(x),$$

where for any point $t \in \mathbb{T}$, we have $\rho(t) \leq t \leq \sigma(t)$.

Third, the discrete form in (1.1) can by transformed to its analogue on time scales as follows: the discrete analogue is

$$\frac{1}{n}\sum_{i=1}^{n} a(i). \tag{3.1}$$

For the discrete case $\mathbb{T} = \mathbb{N} = \{1, 2, 3, \ldots\}$, we have $\sigma(b) = b + 1$, $\rho(b) = b - 1$, and $\mu(b) = 1$; i.e. "$\rho(b) < b < \sigma(b)$".

The form (3.1) can be written as follows:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} a(i) &= \frac{1}{(n+1)-1}\sum_{i=1}^{(n+1)-1} 1 * a(i) \\
&= \frac{1}{\sigma(n)-1}\sum_{i=1}^{\sigma(n)-1} \mu(i)a(i) \\
&= \frac{1}{\sigma(n)-1}\sum_{i\in[1,\sigma(n))} \mu(i)a(i) \\
&= \frac{1}{\sigma(n)-1}\int_1^{\sigma(n)} a(x)\Delta x.
\end{aligned} \tag{3.2}$$

Fourth, when $\mathbb{T} = \mathbb{R}$, then $\rho(t) = t = \sigma(t)$, $\mu(t) = 0$ and the continuous form in (1.2) can be written as follows:

$$\begin{aligned}
\frac{1}{x}\int_0^x f(t)\mathrm{d}t &= \frac{1}{x}\int_0^x f(t)\Delta t \\
&= \frac{1}{\sigma(x)-0}\int_0^{\sigma(x)} f(t)\Delta t.
\end{aligned} \tag{3.3}$$

Therefore, for any $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $x \in [0, \infty)$, we can unify the continuous and discrete forms (3.2) and (3.3) in a single form with a general domain a time scale $\mathbb{T}$. This unified approach is given by

$$\frac{1}{\sigma(x)-\alpha}\int_\alpha^{\sigma(x)} f(t)\Delta t, \quad \text{and} \quad \alpha \in \mathbb{T}.$$

In this case, by following the four steps, we can guess exactly the left side of the dynamic Hardy-type inequality on time scales. After that, by applying the notation and the basics of time scales calculus, we can get the final version on time scales.

Following up on the idea above, in 2005, Řehak [17] established the time scale version of (1.1) and (1.2) and proved that if $\alpha \in \mathbb{T}$, $p > 1$, and f is a nonnegative function such that the delta integral $\int_\alpha^\infty f^p(t)\Delta t$ exists as a finite number, then

$$\int_\alpha^\infty \left(\frac{1}{\sigma(t) - \alpha} \int_\alpha^{\sigma(t)} f(s)\Delta s \right)^p \Delta t < \left(\frac{p}{p-1} \right)^p \int_\alpha^\infty f^p(t)\Delta t, \tag{3.4}$$

unless $f \equiv 0$. In addition, if $(1/t)\,\mu(t) \to 0$ as $t \to \infty$, then the constant $(p/(p-1))^p$ is the best possible. Taking $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and we can get the integral inequality (1.2), while $\mathbb{T} = \mathbb{N}$ with $\sigma(t) = t + 1$, gives the discrete inequality (1.1).

4 Main Results

The aim of this work is to explore various possible generalizations of Hardy-type inequalities on time scales. The idea is to point out to what was recently done in this field by the present author. Some of the following ideas are published and the others are submitted for publications.

We focus on several ideas:

- It is known that the dynamic Hardy-type inequality (3.4) is valid for $p > 1$ "not for $p = 1$". So we aim to get a new reformulation of Hardy-type inequality such that it holds for $p = 1$. This version states that if $\mathbb{T}$ is a time scale with $a, \ell \in \mathbb{T}$, $f \in C_{\mathrm{rd}}([a,\ell]_{\mathbb{T}}, \mathbb{R}^+)$, and either $p \geq 1$ or $p < 0$, then

$$\int_a^\ell \left(\frac{\int_a^x f(y)\Delta y}{x - a} \right)^p \frac{\Delta x}{\sigma(x) - a} \leq \int_a^\ell \left(1 - \frac{\sigma(x) - a}{\ell - a} \right) f^p(x) \frac{\Delta x}{\sigma(x) - a}. \tag{4.1}$$

  Notice that the inequality (4.1) is valid for $p = 1$ and also for the negative parameter $p < 0$.

  Also by using a general version of the Beta function for Hardy-type inequality involving two time scales $\mathbb{T}$ and $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ where $\nu(x) = \frac{x}{y}$, $x, y \in \mathbb{T}$ and $y > \ell$, we assume that $\ell \in \mathbb{T}$, $\ell \geq 0$, $p \geq 1$, $\alpha > 0$ and $f \in C_{\mathrm{rd}}((\ell,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ is a nonincreasing function. Then

$$\int_\ell^\infty \left(\int_x^\infty f(y)\Delta y \right)^p x^{\alpha-1}\Delta x \geq \frac{p}{\alpha} \int_\ell^\infty y^{p+\alpha-1} f^p(y) T(y)\Delta y, \tag{4.2}$$

  where

$$T(y) = \alpha\beta_{\frac{\ell}{y}}(\alpha, p) \; with \; \beta_\lambda(u,v) := \int_\lambda^1 s^{u-1}(1-s)^{v-1}\tilde{\Delta} s \quad \text{for} \quad 0 \leq \lambda < 1.$$

  Here $\beta_0(u,v)$ is the usual β-function $\beta(u,v)$.

- This task is published by Saied et al. [18]. We seek to establish some generalized dynamic inequalities of Hardy-type, with a negative parameter $p < 0$. Assume that a, $b \in \mathbb{T}$, $0 \leq a < b < \infty$, $p < 0$, $r > 1$ and f, $g \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ such that $x/g(x)$ is nondecreasing on (a,b). Then

$$
\begin{aligned}
&\int_a^b g^{-r}(x)F^p(x)\Delta x \\
&\leq \left(\frac{p}{1-r}\right)^{p-1}\int_a^b x^{\frac{1+p-r}{p^*}} f^p(x)\left(1-\left(\frac{x}{b}\right)^{\frac{1-r}{p}}\right)^{p-1}\left(\frac{x}{g(x)}\right)^r\left(\int_a^{\sigma(x)} t^{\frac{1-r}{p}-1}\Delta t\right)\Delta x, \quad (4.3)
\end{aligned}
$$

where $F(x) = \int_x^b f(t)\Delta t$.

To complete the results on time scales for the cases $0 \leq r < 1$ and $r < 0$, by using the chain rule , it is necessary to split it into two cases: $(r-1)/p \geq 1$ and $0 < (r-1)/p \leq 1$.

Assume that $a,\ b \in \mathbb{T}$, $0 \leq a < b < \infty$, $p < 0$, $0 \leq r < 1$ and $f,\ g \in C_{\mathrm{rd}}\left([a,b)_{\mathbb{T}}, \mathbb{R}^+\right)$ such that $\sigma(x)/g(x)$ is nonincreasing on (a,b).
(a) If $(r-1)/p \geq 1$, then

$$
\begin{aligned}
&\int_a^b g^{-r}(x)\left[\Omega(\sigma(x))\right]^p\Delta x \\
&\leq \left(\frac{p}{r-1}\right)^p\int_a^b\left(\frac{x}{\sigma(x)}\right)^{\frac{1-r}{p}}(\sigma(x))^p f^p(x)g^{-r}(x)\left[1-\left(\frac{a}{x}\right)^{\frac{r-1}{p}}\right]^{p-1}\Delta x.
\end{aligned} \tag{4.4}
$$

(b) If $0 < (r-1)/p \leq 1$, then

$$
\int_a^b g^{-r}(x)\left[\Omega(\sigma(x))\right]^p\Delta x \leq \left(\frac{p}{r-1}\right)^p\int_a^b\left(\frac{\sigma(x)}{x}\right)^r(xf(x))^p g^{-r}(x)\left[1-\left(\frac{a}{x}\right)^{\frac{r-1}{p}}\right]^{p-1}\Delta x, \tag{4.5}
$$

where $\Omega(x) = \int_a^x f(t)\Delta t$.

Finally, for the case $r < 0$, we split it into two cases $0 < (r-1)/p \leq 1$ and $(r-1)/p \geq 1$. Let $\mathbb{T}$ be a time scale with $a,\ b \in \mathbb{T}$, $0 \leq a < b < \infty$, $p,\ r < 0$ and $f,\ g \in C_{\mathrm{rd}}\left([a,b)_{\mathbb{T}}, \mathbb{R}^+\right)$ such that $x/g(x)$ is nondecreasing on (a,b).
(a) If $0 < (r-1)/p \leq 1$, then

$$
\int_a^b g^{-r}(x)\left[\Omega(\sigma(x))\right]^p\Delta x \leq \left(\frac{p}{r-1}\right)^p\int_a^b(xf(x))^p g^{-r}(x)\left[1-\left(\frac{a}{x}\right)^{\frac{r-1}{p}}\right]^{p-1}\Delta x. \tag{4.6}
$$

(b) If $(r-1)/p \geq 1$, then

$$
\begin{aligned}
&\int_a^b g^{-r}(x)\left[\Omega(\sigma(x))\right]^p\Delta x \\
&\leq \left(\frac{p}{r-1}\right)^p\int_a^b\left(\frac{x}{\sigma(x)}\right)^{\frac{1-r}{p}+r}(\sigma(x))^p f^p(x)g^{-r}(x)\left[1-\left(\frac{a}{x}\right)^{\frac{r-1}{p}}\right]^{p-1}\Delta x,
\end{aligned} \tag{4.7}
$$

where $\Omega(x) = \int_a^x f(t)\Delta t$.

- This task is published in [19] by Saied et al. For explanation, we establish new Hardy-type inequalities that involve two different spaces ℓ_p and ℓ_η (with $1 < p \leq \eta < \infty$) and a nonnegative general kernel $k(x,t)$.

  Also, we seek to prove some dynamic Hardy-type inequalities involving two different spaces l_p, l_η for $1 < p \leq \eta < \infty$ with a nonnegative general kernel $k(x,t)$ such that for the nonnegative function g, there exists a constant $C > 0$, with

$$\left(\int_a^\infty \left(\int_a^{\sigma(x)} k(x,t)g(t)\Delta t\right)^\eta w(x)\Delta x\right)^{\frac{1}{\eta}} \leq C \left(\int_a^\infty g^p(x)h(x)\Delta x\right)^{\frac{1}{p}}, \tag{4.8}$$

where $a \in \mathbb{T}$, $w, g, h \in CC_{\mathrm{rd}}([a,\infty)_\mathbb{T}, \mathbb{R}^+)$ where g, h are considered to be weighted functions and $k \in C_{\mathrm{rd}}([a,\infty)_\mathbb{T} \times [a,\infty)_\mathbb{T}, \mathbb{R}^+)$ is a kernel function.

The weighted Hardy inequality (4.8) is valid for some $0 < s < 1/p^*$ and $p^* = p/(p-1)$, provided that the weighted functions w and h satisfy that

$$E(s) = \sup_{t\in[a,\infty)_\mathbb{T}} \left(\int_t^\infty k^\eta(x,t)(H(\sigma(x)))^{\eta\left(\frac{1}{p^*}-s\right)} w(x)\Delta x\right)^{\frac{1}{\eta}} (H(\sigma(t)))^s < \infty, \tag{4.9}$$

where $H(x) = \int_a^x h^{1-p^*}(y)\Delta y$ and

$$C \leq \inf_{0<s<\frac{1}{p^*}} \left(\frac{p-1}{p-sp-1}\right)^{\frac{1}{p^*}} E(s).$$

If $1 < p \leq q < \infty$, $0 < s < 1/p^*$, $p^* = p/(p-1)$, g is nonincreasing and h is nondecreasing, then

$$\left[\int_a^\infty \left(\int_a^{\sigma(x)} g(t)\Delta t\right)^\eta w(x)\Delta x\right]^{\frac{1}{\eta}} \leq C \left(\int_a^\infty g^p(x)h(x)\Delta x\right)^{\frac{1}{p}}, \tag{4.10}$$

if and only if

$$E(s) = \sup_{y\in[a,\infty)_\mathbb{T}} (H(\sigma(y)))^s \left[\int_y^\infty (H(\sigma(x)))^{\frac{\eta(p-sp-1)}{p}} w(x)\Delta x\right]^{\frac{1}{\eta}} < \infty, \tag{4.11}$$

where $H(x) = \int_a^x h^{1-p^*}(s)\Delta s$ and the constant C satisfies the estimate

$$\sup_{0<s<\frac{1}{p^*}} \left(\frac{ps}{ps+1}\right)^{\frac{1}{p}} A(s) \leq C \leq \inf_{0<s<\frac{1}{p^*}} \left(\frac{p-1}{p(1-s)-1}\right)^{\frac{1}{p^*}} A(s). \tag{4.12}$$

- We generalize the Hardy-type inequalities by incorporating more general lower and upper limits of integration and general Hardy-Steklov type and Copson-Steklov type operators.

  It is known that the classical Hardy, Copson and Steklov operators are given by

$$(Ff)(\xi) = \int_0^\xi f(t)\mathrm{d}t, \quad (Cf)(\xi) = \int_0^\xi f(t)\psi(t)\mathrm{d}t \quad \text{and} \quad (Sf)(\xi) = \int_{\xi-1}^{\xi+1} f(t)\mathrm{d}t,$$

  respectively. The Hardy-type inequalities will be generalized by adding general functions to the lower and upper limits of integration. On the other hand, the new inequalities include the general operators: the Hardy-Steklov operator F_sf and Copson-Steklov operator C_sf, which are defined by

$$(F_sf)(\xi) = \int_{r(\xi)}^{\nu(\xi)} f(s)\psi(s)\mathrm{d}s, \quad \xi > 0$$

$$(C_sf)(\xi) = \int_{r(\xi)}^{\nu(\xi)} \frac{f(s)\psi(s)}{\Psi(s)}\mathrm{d}s, \quad \xi > 0 \quad \text{and} \quad \Psi(s) = \int_0^s \psi(t)\mathrm{d}t, \quad \xi \in (0,\infty),$$

where f, ψ are nonnegative measurable functions on $(0, \infty)$ and r, ν are increasing differentiable functions on $[0, \infty)$ such that

$$\begin{cases} 0 < r(\xi) < \nu(\xi) < \infty \quad \text{for all} \quad \xi \in (0, \infty), \\ r(0) = \nu(0) = 0 \quad \text{and} \quad r(\infty) = \nu(\infty) = \infty. \end{cases}$$

Thus we seek to establish the time scale version of inequalities involving Hardy-Steklov type operators and Copson-Steklov type operators, to get the continuous and discrete analogues of these inequalities, see [3].

To discover the results on time scales, we assume that $f, \psi, r, \nu \in C_{rd}((0, \epsilon)_{\mathbb{T}}, \mathbb{R}^+)$ such that the following properties hold:

$$\begin{aligned} &r, \nu \quad \text{are} \quad \Delta\text{-differentiable and increasing} \\ &0 < r(\xi) < \nu(\xi) < \infty \quad \text{for all} \quad \xi \in (0, \infty), \\ &r(0) = \nu(0) = 0 \quad \text{and} \quad r(\infty) = \nu(\infty) = \infty. \end{aligned} \tag{4.13}$$

Before we start to state our results, we define the Hardy-Steklov operator $F_s f$ and Copson-Steklov operator $C_s f$ on time scales, which are given by

$$\begin{aligned} (F_s f)(\xi) &= \int_{r(\xi)}^{\nu(\xi)} f(s)\psi(s)\Delta s, \quad \xi > 0 \\ (C_s f)(\xi) &= \int_{r(\xi)}^{\nu(\xi)} \frac{f(s)\psi(s)}{\Phi(s)}\Delta s, \quad \xi > 0 \quad \text{and} \quad \Phi(t) = \int_0^t \psi(s)\Delta s. \end{aligned}$$

The result by the Hardy-Steklov operator $F_s f$, shows that if $\mathbb{T}$ is a time scale with $\epsilon \in \mathbb{T}$, $\epsilon > 0$ and $l, \eta, \alpha \in \mathbb{R}$ with $1 < l \leq \eta < \infty$ and $\alpha < l - 1$, then

$$\begin{aligned} &\int_0^\epsilon \frac{\psi(\xi)}{(\Phi(\sigma(\xi)))^{l-\alpha}} (F_s f)^l(\xi)\Delta\xi \\ &\leq \left(\frac{l}{l-\alpha-1}\right)^l \left(\int_0^\epsilon \psi(\xi)\Delta\xi\right)^{\frac{\eta-l}{\eta}} \left(\int_0^\epsilon \frac{\psi(\xi)}{(\Phi(\sigma(\xi)))^{\eta-\frac{\alpha\eta}{l}}} \left(\frac{K(\xi)(F_s f)^{l-1}(\sigma(\xi))}{(F_s f)^{l-1}(\xi)}\right)^\eta \Delta\xi\right)^{\frac{l}{\eta}}, \end{aligned} \tag{4.14}$$

where

$$K(\xi) = \frac{\Phi(\sigma(\xi))}{\psi(\xi)} \left[\nu^\Delta(\xi) f(\nu(\xi))\psi(\nu(\xi)) - r^\Delta(\xi) f(r(\xi))\psi(r(\xi))\right] > 0.$$

Also for $\epsilon \in \mathbb{T}$, $\epsilon > 0$, $l, \eta > 0$, $\alpha \in \mathbb{R}$, $0 < l < 1$ with $\alpha > l$ and $l > \eta$, we obtain

$$\begin{aligned} &\int_0^\epsilon \frac{\psi(\xi)}{(\Phi(\sigma(\xi)))^{l-\alpha}} (F_s f)^l(\xi)\Delta\xi \\ &\geq l^l \left(\int_0^\epsilon \psi(\xi)\Delta\xi\right)^{\frac{\eta-l}{\eta}} \left(\int_0^\epsilon \frac{\psi(\xi)}{(\Phi(\sigma(\xi)))^{\eta-\frac{\alpha\eta}{l}}} \left(\frac{K(\xi)(F_s f)^{l-1}(\sigma(\xi)) \int_{\sigma(\xi)}^\epsilon \psi(s)\Delta s}{\Phi(\sigma(\xi))(F_s f)^{l-1}(\xi)}\right)^\eta \Delta\xi\right)^{\frac{l}{\eta}}, \end{aligned} \tag{4.15}$$

where

$$K(\xi) = \frac{\Phi(\sigma(\xi))}{\psi(\xi)} \left[\nu^{\Delta}(\xi) f(\nu(\xi))\psi(\nu(\xi)) - r^{\Delta}(\xi) f(r(\xi))\psi(r(\xi))\right] > 0.$$

The result involving Copson-Steklov operator $C_s f$, states that if $\mathbb{T}$ is a time scale with $\epsilon \in \mathbb{T}$, $\epsilon > 0$, $1 < l \leq \eta < \infty$ and $\alpha \in \mathbb{R}$ such that $\alpha < l - 1$, then

$$\begin{aligned}
&\int_0^{\epsilon} \frac{\psi(\xi)}{(\Phi(\sigma(\xi)))^{l-\alpha}} (C_s f)^l(\xi) \Delta\xi \\
&\leq \left(\frac{l}{l-\alpha-1}\right)^l \left(\int_0^{\epsilon} \psi(\xi)\Delta\xi\right)^{\frac{\eta-l}{\eta}} \left(\int_0^{\epsilon} \frac{\psi(\xi)}{(\Phi(\sigma(\xi)))^{\eta-\frac{\alpha\eta}{l}}} \left(\frac{J(\xi)(C_s f)^{l-1}(\sigma(\xi))}{(C_s f)^{l-1}(\xi)}\right)^{\eta} \Delta\xi\right)^{\frac{l}{\eta}},
\end{aligned} \tag{4.16}$$

where

$$J(\xi) = \frac{\Phi(\sigma(\xi))}{\psi(\xi)} \left(\nu^{\Delta}(\xi) \frac{f(\nu(\xi))\psi(\nu(\xi))}{\Phi(\nu(\xi))} - r^{\Delta}(\xi) \frac{f(r(\xi))\psi(r(\xi))}{\Phi(r(\xi))}\right) > 0.$$

- The another aim of Hardy inequality is to derive the boundedness of the following integral

$$\int_0^{\alpha} \left(f(x) - \frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p \mathrm{d}x, \quad p > 1, \quad \alpha > 0,$$

which can be viewed as a global measure of the dispersion of the function f around its integral mean, i.e., $(1/x)\int_0^x f(t)dt$, which gives the standard deviation (L^2 dispersion) when $p = 2$. Our aim is to quantify this deviation using a dynamic inequality approach on time scales.

This version on time scales "the global of the dispersion of the function f around its integral mean", states that if $a, b \in \mathbb{T}$ with $b > a$ and $p > 1$ and $f \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ is a differentiable, nondecreasing function such that $\lim_{t\to a}(t-a)f(t) = 0$, then

$$\begin{aligned}
&\int_a^b \left(f(\sigma(x)) - \frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right)^p \Delta x \\
&\leq \left(\frac{p}{p-1}\right)^p \int_a^b \left((\sigma(x)-a)f^{\Delta}(x)\right)^p \left[\left(\frac{\sigma(x)-a}{x-a}\right)^{\frac{p-1}{p}} - \left(\frac{\sigma(x)-a}{b-a}\right)^{\frac{p-1}{p}}\right] \Delta x.
\end{aligned} \tag{4.17}$$

For the case $p = 2$, we have the term

$$\begin{aligned}
&\int_a^b \left(f(\sigma(x)) - \frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right)^2 \Delta x \\
&= \int_a^b \left(f(\sigma(x)) - \frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right)\left(f(\sigma(x)) - \frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right) \Delta x.
\end{aligned}$$

Following up on the idea above, we present a new version that involve two different functions f and g. Let $a, b \in \mathbb{T}$ with $b > a$, $p > 1$ and let $f, g \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ be differentiable, nondecreasing functions such that $\lim_{t\to a}(t-a)f(t)=0$ and $\lim_{t\to a}(t-a)g(t)=0$. Then

$$\begin{aligned}&\int_a^b\left(f(\sigma(x))-\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)}f(t)\Delta t\right)\left(g(\sigma(x))-\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)}g(t)\Delta t\right)\Delta x\\&\le\frac{p^2}{p-1}\left(\int_a^b\left((\sigma(x)-a)f^\Delta(x)\right)^p\left[\left(\frac{\sigma(x)-a}{x-a}\right)^{\frac{p-1}{p}}-\left(\frac{\sigma(x)-a}{b-a}\right)^{\frac{p-1}{p}}\right]\Delta x\right)^{\frac{1}{p}}\\&\times\left(\int_a^b\left((\sigma(x)-a)g^\Delta(x)\right)^{\frac{p}{p-1}}\left[\left(\frac{\sigma(x)-a}{x-a}\right)^{\frac{1}{p}}-\left(\frac{\sigma(x)-a}{b-a}\right)^{\frac{1}{p}}\right]\Delta x\right)^{\frac{p-1}{p}}.\end{aligned}\tag{4.18}$$

Also, we introduce the next result that involves the term $f(\sigma(x))g(\sigma(x))$ such that the functions f and g are bounded. Assume that $a,b\in\mathbb{T}$ with $b>a$, $p>1$ and $f,g\in C_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R}^+)$ are nondecreasing functions such that $f(b)<\infty$, $g(b)<\infty$, $\lim_{t\to a}(t-a)f(t)=0$ and $\lim_{t\to a}(t-a)g(t)=0$. Then

$$\begin{aligned}&\int_a^b\left|f(\sigma(x))g(\sigma(x))-\frac{1}{(\sigma(x)-a)^2}\left(\int_a^{\sigma(x)}f(t)\Delta t\right)\left(\int_a^{\sigma(x)}g(t)\Delta t\right)\right|^p\Delta x\\&\le 2^{p-1}\max\left[f^p(b),g^p(b)\right]\left(\frac{p}{p-1}\right)^p\\&\times\int_a^b(\sigma(x)-a)^p\left[\left(f^\Delta(x)\right)^p+\left(g^\Delta(x)\right)^p\right]\left[\left(\frac{\sigma(x)-a}{x-a}\right)^{\frac{p-1}{p}}-\left(\frac{\sigma(x)-a}{b-a}\right)^{\frac{p-1}{p}}\right]\Delta x.\end{aligned}\tag{4.19}$$

5 Application

Inequalities—especially integral inequalities—are important in many areas of mathematics. As pointed out in [2, 5], their discrete analogs can be more complicated to prove. To unify continuous and discrete analyses (and thereby avoid repeating similar proofs in various contexts), we turn to the framework of *time scales*, which generalizes the concept of differentiation and integration to arbitrary closed subsets of real numbers (time scales), involving both discrete and continuous cases as special cases. This will allow us unify and generalize these cases to get a general inequality on a time scale $\mathbb{T}$. So we can get either new continuous cases, new discrete cases, new formula in quantum calculus or a general case on a time scale $\mathbb{T}$.

When $\mathbb{T}=\mathbb{R}$, classical integral inequalities from continuous calculus re-emerge. Likewise, when $\mathbb{T}=\mathbb{N}_0$, one recovers the standard discrete inequalities from difference calculus; and when $\mathbb{T}=q^{\mathbb{N}_0}$ for $q>1$, one obtains inequalities in the setting of quantum calculus.

A motivating example is provided by Saker et al. [20], who studied the time-scale version of the classical Hardy-type integral inequality. Specifically, they proved that the dynamic inequality

$$\left(\int_a^b u(x)\left(\int_a^{\sigma(x)}f(t)\Delta t\right)^q\Delta x\right)^{1/q}\le C\left(\int_a^b \upsilon(x)f^p(x)\Delta x\right)^{1/p},\ 1<p\le q<\infty,\tag{5.1}$$

holds for all nonnegative rd-continuous functions f on $[a,b]_{\mathbb{T}}$ with $a,\ b\in\mathbb{T}$ if and only if the following condition holds

$$B=\sup_{a<x<b}\left(\int_x^b u(t)\Delta t\right)^{1/q}\left(\int_a^{\sigma(x)}\upsilon^{1-p^*}(t)\Delta t\right)^{1/p^*}<\infty,\quad p^*=\frac{p}{p-1},\tag{5.2}$$

where u, v are nonnegative rd-continuous weighted functions. Moreover, the estimate for the constant $C>0$ in (5.1) is given by

$$B \le C \le \left(1+\frac{q}{p^*}\right)^{1/q}\left(1+\frac{p^*}{q}\right)^{1/p^*} B.$$

As an application, in the same paper [20], the authors used this Hardy-type inequality to study oscillation properties of the second-order dynamic equation

$$\left(r(t)\varphi_\alpha(x^\Delta)\right)^\Delta + s(t)\varphi_\alpha(x^\sigma) = 0, \tag{5.3}$$

where $1<\alpha<\infty$, $\varphi_\alpha(y) = |y|^{\alpha-2}y$ and $r,s \in C_{rd}\left([a,b]_{\mathbb{T}}, \mathbb{R}\right)$ with $r(t) \neq 0$ for $t \in [a,b]_{\mathbb{T}}$. They showed that (5.3) is nonoscillatory if and only if the Hardy-type inequality (5.1) holds for $p=q=\alpha$ with $C=1$.

6 Numerical Examples

In this section, we present some numerical examples to inform the validity of our results.

First, we begin showing some examples on inequality (4.1). In the continuous case (when $\mathbb{T}=\mathbb{R}$), where $\sigma(x)=x$, the inequality (4.1) reduces its following integral form

$$\int_a^\ell \left(\frac{\int_a^x f(y)\mathrm{d}y}{x-a}\right)^p \frac{\mathrm{d}x}{x-a} \le \int_a^\ell \left(1-\frac{x-a}{\ell-a}\right) f^p(x)\frac{\mathrm{d}x}{x-a}. \tag{6.1}$$

By checking the last inequality when $p=\frac{-1}{2}<0$, $a=0$, $\ell=4$, and $f(x)=x^{\frac{-1}{2}}$, the left side of (6.1) gives

$$\int_a^\ell \left(\frac{\int_a^x f(y)\mathrm{d}y}{x-a}\right)^p \frac{\mathrm{d}x}{x-a} = \int_0^4 \left(\frac{\int_0^x y^{\frac{-1}{2}}\mathrm{d}y}{x}\right)^{\frac{-1}{2}} \frac{\mathrm{d}x}{x} = \frac{1}{\sqrt{2}}\int_0^4 x^{\frac{-3}{4}}\mathrm{d}x = 4 \tag{6.2}$$

and the right side of (6.1) becomes

$$\int_a^\ell \left(1-\frac{x-a}{\ell-a}\right) f^p(x)\frac{\mathrm{d}x}{x-a} = \int_0^4 \left(1-\frac{x}{4}\right) x^{\frac{-3}{4}}\mathrm{d}x = 4.525. \tag{6.3}$$

From (6.2) and (6.3), we find that the integral inequality (6.1) holds for the negative parameter $p<0$.

Second, the discrete form of (4.1) (when $\mathbb{T}=\mathbb{N}$) is given by

$$\sum_{x=a}^{\ell-1} \frac{1}{x+1-a}\left(\frac{\sum_{y=a}^{x-1} f(y)}{x-a}\right)^p \le \sum_{x=a}^{\ell-1} \frac{1}{x+1-a}\left(1-\frac{x+1-a}{\ell-a}\right) f^p(x). \tag{6.4}$$

To check the validity of (6.4) for $\mathbb{T}=\mathbb{N}$, we take $p=-3<0$, $a=1$, $\ell=5$, and $f(y)=\frac{1}{y(y+1)}$, then $\sum_{y=a}^{x-1} f(y) = \sum_{y=1}^{x-1}\frac{1}{y(y+1)} = \frac{x-1}{x}$ and the left side of (6.4) gives

$$\sum_{x=a}^{\ell-1} \frac{1}{x+1-a}\left(\frac{\sum_{y=a}^{x-1} f(y)}{x-a}\right)^p = \sum_{x=1}^{4} x^2 = 30 \tag{6.5}$$

and the right side of (6.4) gives

$$\sum_{x=a}^{\ell-1} \frac{1}{x+1-a}\left(1-\frac{x+1-a}{\ell-a}\right) f^p(x) = \sum_{x=1}^{4} x^2(x+1)^3\left(1-\frac{x}{4}\right) = 204. \tag{6.6}$$

From (6.5) and (6.6), we observe that the discrete inequality (6.4) holds for the negative parameter $p < 0$.

The next example is related to checking the inequality (4.5). For the continuous case (when $\mathbb{T} = \mathbb{R}$) by taking $p = \frac{-1}{2} < 0$, $a = 0$, $b = 1$, $r = \frac{1}{2}$, $f(x) = x^{\frac{-1}{2}}$, and $g(x) = x$ for $x \in (0,1]$, then the left side of (4.5) gives

$$\int_a^b g^{-r}(x)\,[\Omega(\sigma(x))]^p\,\Delta x = \int_a^b g^{-r}(x)\left[\int_a^x f(t)\mathrm{d}t\right]^p \mathrm{d}x = \int_0^1 x^{\frac{-1}{2}}\left[\int_0^x t^{\frac{-1}{2}}\mathrm{d}t\right]^{\frac{-1}{2}} \mathrm{d}x = 2.8284 \tag{6.7}$$

and the right side of (4.5) gives

$$\begin{aligned} &\left(\frac{p}{r-1}\right)^p \int_a^b \left(\frac{\sigma(x)}{x}\right)^r (xf(x))^p\, g^{-r}(x)\left[1-\left(\frac{a}{x}\right)^{\frac{r-1}{p}}\right]^{p-1} \Delta x \\ &= \left(\frac{p}{r-1}\right)^p \int_a^b (xf(x))^p\, g^{-r}(x)\left[1-\left(\frac{a}{x}\right)^{\frac{r-1}{p}}\right]^{p-1} \mathrm{d}x \\ &= \int_0^1 x^{\frac{-3}{4}}\mathrm{d}x \\ &= 4. \end{aligned} \tag{6.8}$$

From (6.7) and (6.8), we observe that the inequality (4.5) holds for the continuous case.

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